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Noncommutative Spacetimes

Symmetries in Noncommutative Geometry and Field Theory

Springer

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Preface

There are many approaches to noncommutative geometry and its use in physics, the operator algebra and C^*-algebra one, the deformation quantization one, the quantum group one, and the matrix algebra/fuzzy geometry one. This volume introduces and develops the subject by presenting in particular the ideas and methods recently pursued by Julius Wess and his group.

These methods combine the deformation quantization approach based on the notion of star product and the deformed (quantum) symmetries methods based on the theory of quantum groups. The merging of these two techniques has proven very fruitful in order to formulate field theories on noncommutative spaces. The aim of the book is to give an introduction to these topics and to prepare the reader to enter the research field himself/herself. This has developed from the constant interest of Prof. W. Beiglboeck, editor of LNP, in this project, and from the authors experience in conferences and schools on the subject, especially from their interaction with students and young researchers.

In fact quite a few chapters in the book were written with a double purpose, on the one hand as contributions for school or conference proceedings and on the other hand as chapters for the present book. These are now harmonized and complemented by a couple of contributions that have been written to provide a wider background, to widen the scope, and to underline the power of our methods.

The different chapters however remain essentially self-consistent and can be read independently. Subject to the individual interests of the reader they can be grouped by topic: noncommutative gauge theory (Chaps. 1, 2, 4, 5), noncommutative gravity (Chaps. 1, 3, 8), and noncommutative geometry and quantum groups (Chaps. 6, 7, 9). This very structure of the book took definite shape a little more than a year ago, at the Alessandria conference "Noncommutative Spacetime Geometries" in March 2007, where all the authors met. At the Bayrishzell workshop "On Noncommutativity and Physics" in May 2007 the order of the chapters was then finalized.

The order of the chapters is "physics first"; the mathematics follows the physical motivations in order to strengthen the physical intuition and investigations and to provide a sharpening of the mathematical methods. These is turn are then used for further physical developments. Accordingly the book is divided into a more physical first part and a more mathematical second part, although the division is not sharp, physical applications being considered in the second part too.

The first chapter is an introduction and an overview. The reader encounters the notion of star product and is introduced to the differential calculus on noncommutative spaces and to the deformed Lie algebras (twisted Hopf algebras) of gauge transformations and diffeomorphisms. The second chapter develops in more detail deformed gauge theories. Pedagogic examples with matter fields are also presented. The third chapter discusses in the same spirit the deformed algebra of differential operators and hence a deformation of the theory of gravity. Changes to the original text of Julius Wess mainly appear in the added footnotes and in the added Appendix 1.9.

The fourth chapter is a comparison between two approaches to noncommutative gauge theory, the twisted gauge theory approach (based on deformed Lie algebras) and the Seiberg–Witten approach.

Field theories can be studied also on more general noncommutative spaces, not just on the Moyal–Weyl one characterized by the $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$ noncommutative relations among coordinates (with $\theta^{\mu\nu}$ constant). Chapter 5 describes the case of κ-deformed spacetime.

Part II of the book opens with a chapter on the basics of noncommutative manifolds in the C^*-algebraic approach, the guiding example being the quantum mechanical phase space, i.e., the Moyal–Weyl noncommutative space. Quantum groups (noncommutative manifolds with a group structure) are then studied in Chap. 7. Their quantum Lie algebras are also studied, quantum Lie algebras being the underlying symmetries of field theories on noncommutative spaces. Chapter 8 complements Chap. 3 and studies noncommutative geometries obtained by deforming commutative geometries via a twist. These geometries have twisted symmetries (twisted quantum group symmetries). Twisted diffeomorphisms lead to a noncommutative theory of gravity.

While twisting of spacetime symmetries leads to deformed field theories, twisting of dynamical symmetries leads to new (deformed) quantum integrable systems. The last chapter deals with this other application of twisted symmetries. In a sense this chapter closes a circle, we deform field theories by considering noncommutative spacetimes. These are obtained via a twist procedure. We recognize and exploit the underlying twisted and quantum group symmetries. These structures first occurred in $1+1$-dimensional quantum integrable systems; the twist procedure can be also applied in this context and leads to new physical systems.

A final chapter has later been added and describes the contributions of Julius Wess to noncommutative geometry. As can be inferred from his joint works he was able to enroll many students and collaborators in his research projects. This was due to his scientific charisma, always downplayed, and to the easiness in relating with colleagues and younger collaborators, a characteristic aspect of his personality.

Julius Wess was extremely active until his last day, his constant passion for research was so strongly conveyed that concentration and energy for advancing in the research were multiplied. In his vision the main aims and questions were always in the foreground, progress was constant, in many little steps, like that patient walking pace you keep when aiming at the very top. We miss his encouragement, hints, and judgments and that very state of searching together that empowered our discovering abilities. We hope the reader can experience his calm impetus along with the formulae in this book, and thus be more easily brought to the research frontiers of this field to be further developed.

Paolo Aschieri, Marija Dimitrijević, Petr Kulish and Fedele Lizzi

Alessandria, October 2008

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Contents

Chapter 1
Differential Calculus and Gauge Transformations on a Deformed Space

Julius Wess

Deformed gauge transformations on deformed coordinate spaces are considered for any Lie algebra. The representation theory of this gauge group forces us to work in a deformed Lie algebra as well. This deformation rests on a twisted Hopf algebra, thus we can represent a twisted Hopf algebra on deformed spaces. That leads to the construction of Lagrangian invariant under a twisted Lie algebra.

1.1 Introduction

Since Newton the concept of space and time has gone through various changes. All stages, however, had in common the notion of a continuous linear space. Today we formulate fundamental laws of physics, field theories, gauge field theories, and the theory of gravity on differentiable manifolds. That a change in the concept of space for very short distances might be necessary was already anticipated in 1854 by Riemann in his famous inaugural lecture [1]:

> Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena...
>
> ...An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to ensure that this work is not hindered by too restricted concepts, and that the progress in comprehending the connection of things is not obstructed by traditional prejudices.

There are indications today that at very short distances we might have to go beyond differential manifolds.

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In contrast to coordinate space, phase space – the space of coordinates and momenta – has seen a more dramatic change. Forced by quantum mechanics we understand it as an algebraic entity based on Heisenberg's commutation relations for canonical variables

$$[x^i, p^j] = i\hbar\delta^{ij},$$
$$[x^i, x^j] = 0, \qquad [p^i, p^j] = 0. \tag{1.1}$$

Space and momenta have become noncommutative, they form an algebra.

This algebraic setting has proved to be extremely successful. We would not understand fundamental facts of physics, the uncertainty relation, or the existence of atoms, e.g., without it.

The uncertainty relation, however, brings us in conflict with Einstein's law of gravity if we assume continuity in the space variable for arbitrary small distances [2]. From the uncertainty relation

$$\Delta x^i \cdot \Delta p^j \geq \frac{\hbar}{2}\delta^{ij} \tag{1.2}$$

follows that we need very high energies to measure very short distances. High energies lead to the formation of black holes with a Schwarzschild radius proportional to the energy. In turn, this does not allow the measurement of distances smaller than the Schwarzschild radius.

This is only one of several arguments that we have to expect some changes in physics for very small distances. Other arguments are based on the singularity problem in Quantum field theory and the fact that Einstein's theory of gravity is nonrenormalizable when quantized [2].

Why not try an algebraic concept of spacetime that could guide us to changes in our present formulation of laws of physics? This is different from the discovery of quantum mechanics. There physics data forced us to introduce the concept of noncommutativity. Now we take noncommutativity as a guide into an area of physics where physical data are almost impossible to obtain. We hope that it might solve some conceptual problems that are still left at very small distances. We also hope that it could lead to predictions that can be tested in not too far a future by experiment.

The idea of noncommutative coordinates is almost as old as quantum field theory. Heisenberg proposed it in a letter to Peierls [3] to solve the problem of divergent integrals in relativistic quantum field theory. The idea propagated via Pauli to Oppenheimer. Finally H. S. Snyder, a student of Oppenheimer, published the first systematic analysis of a quantum theory built on noncommutative spaces [4]. Pauli called this work mathematically ingenious but rejected it for reasons of physics [5].

In the meantime the theory of renormalization has found a reasonable answer to the divergency problem in quantum field theory. We should not forget, however, that it was the renormalization problem that led to quantum gauge theories and to supersymmetric theories. Only Einstein's theory of gravity remained unrenormalizable when quantized.

From quantum spaces and quantum groups new mathematical concepts have emerged by the pioneering work of V. G. Drinfel'd, L. Faddeev, M. Jimbo and I. Manin [6–9]. This also revived the interest in noncommutativity in physics.

Flato and Sternheimer [10, 11] have developed the machinery of deformation quantization. There noncommutativity appears in the form of noncommutative products of functions of commutative variables. These products are called star products ($\star$-products). They deform the commutative algebras of functions based on pointwise multiplication to noncommutative algebras based on the star product.

Deformation theory has reached a very high and powerful level by the work of Kontsevich and his formality theorem [12].

These developments make it worthwhile to reexamine the concept of noncommutative coordinates in physics. We first show that the points of view of noncommutative coordinates and of noncommutative $\star$-products are intimately related.

1.2 The algebra

It is the algebraic structure of continuous spaces that we want to deform. To show this structure we first consider polynomials in commutative variables $x^1,\dots,x^N$ with complex coefficients. To define them we first define the algebra over $\mathbb{C}$, freely generated by the variables $x^1,\dots,x^N$

$$\mathbb{C}[x^1,\dots,x^N]. \tag{1.3}$$

This means that we take all the finite formal products of the N elements $x^1,\dots,x^N$ as a basis for a linear space over $\mathbb{C}$. A different ordering in the coordinates gives rise to an independent element of the basis! Multiplication of the basis elements is natural. The unit 1 in the algebra is the unique basis element of zero degree. This then defines the freely generated algebra.[1]

Next we consider the relations

$$\mathscr{R}_x : \qquad [x^\mu, x^\nu] = 0. \tag{1.4}$$

They generate a two-sided ideal (left and right) in $\mathbb{C}[x^1,\dots,x^N]$. The quotient

$$\mathscr{P}_x = \frac{\mathbb{C}[x^1,\dots,x^N]}{I_{\mathscr{R}}} \tag{1.5}$$

is the algebra of polynomials in N commuting variables. The definition of the algebra $\mathscr{P}_x$ can be extended. First the algebra $\mathbb{C}[x^1,\dots,x^N]$ is extended by including a parameter h and by considering the algebra of formal power series in h with coefficients in $\mathbb{C}[x^1,\dots,x^N]$. This algebra is denoted by

[1] In this book we will always consider associative algebras. Therefore, in the sequel the term algebra will always refer to an associative algebra.

$$\mathbb{C}[x^1,\ldots,x^N][[h]]. \tag{1.6}$$

Then we consider the ideal in $\mathbb{C}[x^1,\ldots,x^N][[h]]$ generated by the relations (1.4). The quotient

$$\mathscr{A}_x = \frac{\mathbb{C}[x^1,\ldots,x^N][[h]]}{I_{\mathscr{R}}} \tag{1.7}$$

is the sought extension of $\mathscr{P}_x$.

Up to now we have used algebraic concepts only. No topological properties have been mentioned. Our ambition is to go as far as possible in developing a deformed differential calculus without invoking topological properties. This can be done by considering formal power series in h.

A natural way is to deform the relation (1.4)[2]:

$$\hat{\mathscr{R}}: \qquad [\hat{x}^\mu,\hat{x}^\nu] - ihC^{\mu\nu}(\hat{x}) = 0, \tag{1.8}$$

where $C^{\mu\nu}(\hat{x}) \in \mathbb{C}[\hat{x}^1,\ldots,\hat{x}^N][[h]]$. For $h=0$ we obtain the usual algebra of commuting variables as introduced above.

The relations (1.8) generate a two-sided ideal $I_{\hat{\mathscr{R}}}$: the linear span of elements

$$(\hat{x}\ldots\hat{x})\Big([\hat{x}^\mu,\hat{x}^\nu] - ihC^{\mu\nu}(\hat{x})\Big)(\hat{x}\ldots\hat{x}), \tag{1.9}$$

where $(\hat{x}\ldots\hat{x})$ stands for an arbitrary product of $\hat{x}$ in the freely generated algebra $\mathbb{C}[\hat{x}^1,\ldots,\hat{x}^N][[h]]$. Multiplying an element of $I_{\hat{\mathscr{R}}}$ by an element of $\mathbb{C}[\hat{x}^1,\ldots,\hat{x}^N][[h]]$ from the right or left yields an element of $I_{\hat{\mathscr{R}}}$ again. The quotient

$$\hat{\mathscr{A}}_{\hat{x}} = \frac{\mathbb{C}[\hat{x}^1,\ldots,\hat{x}^N][[h]]}{I_{\hat{\mathscr{R}}}} \tag{1.10}$$

is an algebra in the noncommuting coordinates $\hat{x}$.

Well-known examples of such algebras are as follows:

1. The deformation with $\hat{x}$-independent constant $C^{\mu\nu}$. This is the same algebra in coordinate space as the Heisenberg algebra in phase space. We will call it the canonical or for short θ-deformation

$$[\hat{x}^\mu,\hat{x}^\nu] = ih\theta^{\mu\nu} , \tag{1.11}$$

   where $C^{\mu\nu}(\hat{x}) = \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R}$.
2. The Lie algebra type of deformation. In this case $C^{\mu\nu}(\hat{x})$ is linear in the $\hat{x}$-variables,

$$[\hat{x}^\mu,\hat{x}^\nu] = ihf^{\mu\nu}_\rho\hat{x}^\rho . \tag{1.12}$$

[2] In order to stress that these resulting variables are no longer commutative, we denote them with a hat.

The Lie algebra structure constants are $hf_\rho^{\mu\nu}$ and (as always in this book) sum over repeated indices is understood. The algebra $\hat{\mathscr{A}}_{\hat{x}}$ that we are constructing is the universal enveloping algebra of the Lie algebra (1.12).
A particularly interesting example of the Lie algebra type is

$$[\hat{x}^\mu, \hat{x}^\nu] = i(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu), \tag{1.13}$$

with real parameters a^μ. In a basis where $a^i = 0$ for $i \neq N$ and $a^N = 1/\kappa$ we can identify this algebra with the algebra of the κ-deformations [13, 14] (historically for these deformations the parameter κ rather than h is used).

The "size" of the algebra $\hat{\mathscr{A}}_{\hat{x}}$ will depend on the ideal $I_{\hat{\mathscr{R}}}$. It can range from $\mathbb{C}$ to the freely generated algebra itself.[3] We certainly would like an infinite algebra, if possible of the "size" of the algebra $\mathscr{A}_x$ of commuting variables. To be more precise, the vector space of the algebra $\mathscr{A}_x$ can be decomposed into subspaces V_r spanned by monomials of a given degree r. These vector spaces are finite dimensional. A basis of V_r is given by the monomials $x^{i_1}x^{i_2}\ldots x^{i_r}$ with $i_1 \leq i_2 \leq \ldots i_r$. Consider the vector space $F_r = \bigoplus_{s=0}^r V_s$ spanned by all monomials up to degree r. Then we require the vector space $\hat{F}_r$ in $\hat{\mathscr{A}}_{\hat{x}}$ of all polynomials up to degree r in the noncommutative variables to have the same dimension as F_r. We also require the ordered monomials up to degree r

$$\hat{x}^{i_1}\hat{x}^{i_2}\ldots\hat{x}^{i_s}\,, \qquad\qquad i_1 \leq i_2 \leq \ldots i_s\,, \quad 0 \leq s \leq r \tag{1.14}$$

to be a basis of $\hat{F}_r$. We could also consider a different ordering. More in general we require monomials up to degree r, and ordered with respect to any given ordering, to form a basis of $\hat{F}_r$. Thus any monomial of degree r can be rewritten as an ordered polynomial of degree up to r.

When an algebra $\hat{\mathscr{A}}_{\hat{x}}$ satisfies these conditions we say that it has the Poincaré–Birkhoff–Witt (PBW) property. The θ-deformation and the enveloping Lie algebras have this property (this is the PBW theorem).

The art of the game now is to find relations (1.8) that imply the PBW property. This restricts the $\hat{x}$ dependence of $C^{\mu\nu}(\hat{x})$. It is natural to consider $C^{\mu\nu}(\hat{x})$ at most quadratic in $\hat{x}$ and antisymmetric in μ and ν (otherwise we introduce the relations $C_{ij}(\hat{x}) + C_{ji}(\hat{x}) = 0$ in $\mathbb{C}[\hat{x}^1,\ldots,\hat{x}^N][[h]]$, these may lead to $\dim\hat{F}_2 < \dim F_2$).

To be consistent with the reality property $(x^\mu)^* = x^\mu$ we demand a conjugation for $\hat{x}$ as well

$$(\hat{x}^\mu)^* = \hat{x}^\mu, \quad (\hat{x}^\mu\hat{x}^\nu)^* = (\hat{x}^\nu)^*(\hat{x}^\mu)^*, \quad (i)^* = -i. \tag{1.15}$$

This implies

$$(C^{\mu\nu})^* = -C^{\nu\mu} = C^{\mu\nu}. \tag{1.16}$$

[3] In the remaining part of this page and in the following one we omit the formal parameter h. To reinsert it just consider $\hat{\mathscr{A}}_{\hat{x}}$ as algebra over $\mathbb{C}[[h]]$, formal power series in h with coefficients in $\mathbb{C}$ (see also end of Appendix 1.9).

1.3 The star product

In an algebra $\hat{\mathscr{A}}_{\hat{x}}$ with the PBW property, the set of all monomials ordered with respect to a given fixed ordering forms a basis. The symmetric ordering gives fully symmetrized monomials, it is a natural choice but not the only one. The linear span of the basis elements of degree r defines the vector space $\hat{V}_r$. By construction this space has the same dimension as the vector space V_r of polynomials of degree r in N commuting variables.

In this section we extend the vector space isomorphism

$$\hat{V}_r \sim V_r \tag{1.17}$$

to an algebra isomorphism

$$\hat{\mathscr{A}}_{\hat{x}} \sim \mathscr{A}_x^\star, \tag{1.18}$$

as vector spaces $\mathscr{A}_x$ and $\mathscr{A}_x^\star$ coincide. The $\star$-product (or Moyal product) in $\mathscr{A}_x^\star$ is defined so that the algebras $\hat{\mathscr{A}}_{\hat{x}}$ and $\mathscr{A}_x^\star$ are isomorphic. By the vector space isomorphism we map polynomials

$$p(x) \longleftrightarrow \hat{p}(\hat{x}), \tag{1.19}$$

by a map of the basis. Two polynomials $\hat{p}_1(\hat{x})$ and $\hat{p}_2(\hat{x})$ can be multiplied

$$\hat{p}_1(\hat{x}) \cdot \hat{p}_2(\hat{x}) = \widehat{p_1 p_2}(\hat{x}). \tag{1.20}$$

By the isomorphism of (1.19) we map this polynomial back to a polynomial in $\mathscr{A}_x$

$$\widehat{p_1 p_2}(\hat{x}) \mapsto p_1(x) \star p_2(x). \tag{1.21}$$

This defines the star product of two polynomial functions. It is bilinear and associative but noncommutative.

For the θ-deformation in the symmetric basis we obtain [16, 17], see Appendix 1.9 for details,

$$p_1(x) \star p_2(x) = \mu\left(e^{\frac{i}{2}h\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} p_1(x) \otimes p_2(x)\right), \tag{1.22}$$

where μ is the multiplication map

$$\mu\left(f(x) \otimes g(x)\right) = f(x) \cdot g(x). \tag{1.23}$$

This $\star$-product is the well-known Moyal product. It can be extended to C^∞ functions, remaining bilinear and associative. The power series in h will not converge for arbitrary C^∞ functions, we in general consider it as a formal power series.

When we expand in h we obtain

$$f(x) \star g(x) = f(x)g(x) + \mathscr{O}(h)$$

and

$$f(x) \star g(x) - g(x) \star f(x) \\ = \frac{ih}{2}\theta^{\rho\sigma}\Big((\partial_\rho f(x))(\partial_\sigma g(x)) - (\partial_\rho g(x))(\partial_\sigma f(x))\Big) + \mathscr{O}(h^2). \tag{1.24}$$

Equation (1.24) defines the Poisson structure

$$\{f,g\} = \frac{i}{2}\theta^{\rho\sigma}\Big((\partial_\rho f)(\partial_\sigma g) - (\partial_\rho g)(\partial_\sigma f)\Big). \tag{1.25}$$

Kontsevich has shown that for any Poisson structure on a differentiable manifold M there exists a $\star$-product deformation[4] of the algebra $C^\infty(M)[[h]]$ of formal power series in h of smooth functions $C^\infty(M)$ from M to $\mathbb{C}$. The Poisson structure is defined as in (1.24) and (1.25).

Knowing this, it seems natural to investigate noncommutative spaces in the $\star$-product framework.

Our aim now is to formulate laws of physics on an algebra of functions whose product is not the pointwise product but a noncommutative star product. We call this algebra $\mathscr{A}_x^\star$.

One important step in this direction is the development of a differential calculus on this deformed algebra of functions $\mathscr{A}_x^\star$. This we will do next. But let me for the convenience of the reader summarize the notation first. This notation will be systematically used in the first part of the book.

Notation

- $\mu(f \otimes g) = f \cdot g$ – pointwise multiplication.
- $\mu_\star(f \otimes g) = f \star g$ – star multiplication. In the canonical (θ-deformed) case

$$\mu_\star(f \otimes g) = f \star g = \mu\left(e^{\frac{i}{2}h\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\right) \\ = \sum_{n=0}^{\infty}\left(\frac{ih}{2}\right)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\ldots\partial_{\rho_n} f\Big)\Big(\partial_{\sigma_1}\ldots\partial_{\sigma_n} g\Big).$$

- $\mathscr{P}_x$ – algebra of polynomials in N commuting variables $x_1,\ldots,x_N$.
- $\hat{\mathscr{P}}_{\hat{x}}$ – algebra of polynomials in N noncommuting variables $\hat{x}_1,\ldots,\hat{x}_N$.
- $\mathscr{A}_x$ – algebra of formal power series in h of polynomials in N commuting variables $x_1,\ldots,x_N$. Depending from the context, also a completion of it, like for example the algebra of formal power series in h of smooth functions from $\mathbb{R}^N$ to $\mathbb{C}$.

[4] A $\star$-product $\star : C^\infty(M)[[h]] \times C^\infty(M)[[h]] \to C^\infty(M)[[h]]$ is a bidifferential operator (a differential operator on both of its arguments) that is associative, that satisfies $f \star 1 = 1 \star f = f$ for any f, and that at zeroth order in h reduces to the usual commutative product of functions.

In order to stress that higher derivatives may appear in a differential operator (and do appear in $\star$-products) we frequently refer to differential operators as higher order differential operators.

- $\mathscr{A}_{\hat{x}}$ – algebra of formal power series in h of polynomials in N noncommuting variables $\hat{x}_1,\ldots,\hat{x}_N$ or, depending on the context, a completion of it.
- $\mathscr{A}_x^\star$ – algebra of formal power series in h of polynomials in N commuting variables $x_1,\ldots,x_N$ with $\star$-product multiplication. Depending from the context, also a completion of it.
- V_r – linear subspace of $\mathscr{A}_x$ spanned by monomials in $x_1,\ldots,x_N$ of degree r.
- $\hat{V}_r$ – linear subspace of $\mathscr{A}_{\hat{x}}$ spanned by monomials in $\hat{x}_1,\ldots,\hat{x}_N$ of degree r.
- $\mathscr{F}_x$ – linear space of functions in N commuting variables $x_1,\ldots,x_N$.
- $\mathscr{D}_{\{d\}}$ – higher order differential operator acting on $\mathscr{A}_x$.
- $\mathscr{D}^\star_{\{d\}}$ – higher order differential operator acting on $\mathscr{A}_x^\star$.
- $\mathscr{A}\mathscr{D}_{\{d\}}$ – algebra of higher order differential operators $\mathscr{D}_{\{d\}}$.
- $\mathscr{A}^\star\mathscr{D}^\star_{\{d\}}$ – algebra of higher order differential operators $\mathscr{D}^\star_{\{d\}}$.

1.4 A deformed differential calculus

Let us first define a derivative as a map of C^∞ functions to C^∞ functions

$$\begin{aligned}\partial_\mu : \qquad & \mathscr{A}_x \to \mathscr{A}_x\\ f(x) &\mapsto (\partial_\mu f(x)),\end{aligned} \tag{1.26}$$

where $\partial_\mu = \partial/\partial x^\mu$. For polynomials this map can be defined purely algebraically by stating the rule

$$\partial_\mu : \qquad x^\rho \mapsto \delta_\mu^\rho \tag{1.27}$$

and using linearity and the Leibniz rule

$$(\partial_\mu(p_1\cdot p_2)) = (\partial_\mu p_1)\cdot p_2 + p_1\cdot(\partial_\mu p_2). \tag{1.28}$$

We know that this defines the derivative of polynomials. This can be easily extended to formal power series. We use it to define the derivative on $\mathscr{A}_x^\star$ by first mapping an element of $\mathscr{A}_x^\star$ to $\mathscr{A}_x$, differentiate this element in $\mathscr{A}_x$ and map it back to $\mathscr{A}_x^\star$. Thus, we define

$$\begin{aligned}\partial_\mu^\star : \qquad & \mathscr{A}_x^\star \to \mathscr{A}_x^\star\\ f(x)\in\mathscr{A}_x^\star \mapsto f(x)\in\mathscr{A}_x &\mapsto (\partial_\mu f(x))\in\mathscr{A}_x \mapsto (\partial_\mu^\star f(x))\in\mathscr{A}_x^\star.\end{aligned} \tag{1.29}$$

Since the vector space structure of $\mathscr{A}_x$ and $\mathscr{A}_x^\star$ is the same (we denoted it by $\mathscr{F}_x$) the partial derivative $\partial_\mu^\star$ defined in (1.29) coincides with ∂_μ. We could simply (as we frequently do) omit the $\star$ in $\partial_\mu^\star$. The notation $\partial_\mu^\star$ is to emphasize that the partial derivative acts on the deformed algebra of functions $\mathscr{A}_x^\star$. Indeed the Leibniz rule with respect to the $\star$-product changes. In general we have

$$\partial_\mu^\star(f\star g) = (\partial_\mu^\star f)\star g + f\star(\partial_\mu^\star g) + f(\partial_\mu^\star \star)g. \tag{1.30}$$

We have obtained a differential calculus on the deformed algebra of functions $\mathscr{A}_x^\star$.

In the case of θ-deformation the $\star$-operation is x-independent and we obtain the usual Leibniz rule:

$$\partial_\mu^\star(f\star g) = (\partial_\mu^\star f)\star g + f\star(\partial_\mu^\star g). \tag{1.31}$$

For x-dependent $\star$-products, e.g., the κ-deformation, this will change. In the remaining sections we consider the x-independent $\star$-product of the θ-deformation.

1.5 A deformed algebra of differential operators

The differential calculus can be extended to the algebra of higher order differential operators. This algebra includes also functions.

On the commutative algebra $\mathscr{A}_x$ a higher order differential operator is defined as follows

$$\mathscr{D}_{\{d\}} = \sum_{r\geq 0} d_r^{\rho_1\dots\rho_r}(x)\frac{\partial}{\partial x^{\rho_1}}\cdots\frac{\partial}{\partial x^{\rho_r}}. \tag{1.32}$$

These operators form an algebra $\mathscr{A}\mathscr{D}_{\{d\}}$ because we know how to multiply them. This algebra is noncommutative, the commutation relations between coordinates and partial derivatives are $\partial_\mu x^\nu - x^\nu\partial_\mu = \delta_\mu^\nu$; the commutation relations between functions and partial derivatives are

$$\partial_\mu f = (\partial_\mu f) + f\partial_\mu, \quad f\in\mathscr{A}_x. \tag{1.33}$$

We can also write $\partial_\mu^\star x^\nu - x^\nu\partial_\mu^\star = \delta_\mu^\nu$ and

$$\partial_\mu^\star f = (\partial_\mu^\star f) + f\partial_\mu^\star, \quad f\in\mathscr{A}_x^\star. \tag{1.34}$$

For the deformed space of functions we denote the differential operators as

$$\mathscr{D}_{\{d\}}^\star = \sum_{r\geq 0} d_r^{\rho_1\dots\rho_r}(x)\partial_{\rho_1}^\star\dots\partial_{\rho_r}^\star. \tag{1.35}$$

They act on a function as $(\mathscr{D}_{\{d\}}^\star\star f) = \sum_{r\geq 0} d_r^{\rho_1\dots\rho_r}(x)\star\left(\partial_{\rho_1}^\star\dots\partial_{\rho_r}^\star f\right)$. From the Leibniz rule (1.34) and the definition of the $\star$-product for functions we learn how to multiply these deformed operators and in this way obtain the deformed algebra of differential operators $\mathscr{A}^\star\mathscr{D}_{\{d\}}^\star$ acting on elements of the deformed algebra of functions.[5]

[5] The multiplication of the operators $\mathscr{D}^\star$ and $\mathscr{D}'^\star$ is their composition, we denote it by $\mathscr{D}^\star\star\mathscr{D}'^\star$, $\left((\mathscr{D}^\star\star\mathscr{D}'^\star)\star f\right) := (\mathscr{D}^\star\star(\mathscr{D}'^\star\star f))$. For example, the product of the zeroth-order differential operators $\mathscr{D}^\star = d$ and $\mathscr{D}'^\star = d'$ is the zeroth-order differential operator $\mathscr{D}^\star\star\mathscr{D}'^\star = d\star d'$, indeed $(d\star d')\star f = d\star d'\star f = (\mathscr{D}\star(\mathscr{D}'\star f))$. We see that in this case the composition of operators corresponds to the $\star$-product of functions. For first-order differential operators $\mathscr{D}^\star = d^\rho\partial_\rho^\star$ and $\mathscr{D}'^\star = d'^\sigma\partial_\sigma^\star$ we have $\mathscr{D}^\star\star\mathscr{D}'^\star = d^\rho\star(\partial_\rho d'^\sigma)\partial_\sigma^\star + (d^\rho\star d'^\sigma)\partial_\rho^\star\partial_\sigma^\star$.

There is a formal isomorphism between the algebras $\mathscr{A}\mathscr{D}_{\{d\}}$ and $\mathscr{A}^\star\mathscr{D}^\star_{\{d\}}$. We are going to show this for special subalgebras.

There exists a higher order differential operator $X^\star_f \in \mathscr{A}^\star\mathscr{D}^\star_{\{d\}}$ such that[6]

$$X^\star_f \star g = f \cdot g \tag{1.36}$$

where f is an element of $\mathscr{A}_x$ (a zeroth-order differential operator) and on the right-hand side we have its undeformed action on the function g. To find $X^\star_f$ we proceed as follows

$$\begin{aligned} f \cdot g &= \mu\Big(e^{\frac{ih}{2}\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta} e^{-\frac{ih}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} f \otimes g\Big) \\ &= \mu\Big(e^{\frac{ih}{2}\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta} \sum_{r=0}^{\infty}\Big(-\frac{ih}{2}\Big)^r \frac{1}{r!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_r\sigma_r}\big(\partial_{\rho_1}\dots\partial_{\rho_r} f\big)\otimes\big(\partial_{\sigma_1}\dots\partial_{\sigma_r} g\big)\Big) \\ &= \sum_{r=0}^{\infty}\Big(-\frac{ih}{2}\Big)^r \frac{1}{r!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_r\sigma_r}\Big(\partial_{\rho_1}\dots\partial_{\rho_r} f\Big)\star\Big(\partial_{\sigma_1}\dots\partial_{\sigma_r} g\Big). \end{aligned} \tag{1.37}$$

The operator we are looking for is

$$X^\star_f = \sum_{r=0}^{\infty}\Big(-\frac{ih}{2}\Big)^r \frac{1}{r!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_r\sigma_r}\Big(\partial_{\rho_1}\dots\partial_{\rho_r} f\Big)\partial^\star_{\sigma_1}\dots\partial^\star_{\sigma_r}\,. \tag{1.38}$$

It is a higher order differential operator acting on $\mathscr{F}_x$.

Because $f \cdot g$ is again an element of $\mathscr{F}_x$ we can act with $X^\star_h$ on it

$$h\cdot f\cdot g = (h\cdot f)\cdot g = X^\star_{(hf)}\star g = h\cdot(f\cdot g) = h(X^\star_f\star g) = X^\star_h\star(X^\star_f\star g). \tag{1.39}$$

It follows that $X^\star_f$ represents the algebra $\mathscr{A}_x$

$$X^\star_g \star X^\star_f = X^\star_{gf}. \tag{1.40}$$

Let us consider vector fields

$$\xi = \xi^\mu(x)\partial_\mu. \tag{1.41}$$

Their product is again in $\mathscr{A}\mathscr{D}_{\{d\}}$

$$\xi\eta = \xi^\mu(x)\big(\partial_\mu\eta^\rho(x)\big)\partial_\rho + \xi^\mu(x)\eta^\rho(x)\partial_\mu\partial_\rho. \tag{1.42}$$

Through the Lie bracket the vector fields form an algebra

$$\begin{aligned} [\xi,\eta] &= \Big(\xi^\mu(\partial_\mu\eta^\rho) - \eta^\mu(\partial_\mu\xi^\rho)\Big)\partial_\rho \\ &= (\xi\times\eta)^\rho\partial_\rho = \xi\times\eta. \end{aligned} \tag{1.43}$$

[6] We should write $(X^\star_f\star g) = f\cdot g$ in order to stress that $X^\star_f$ acts on the function g. Since we never consider the product of the differential operators $X^\star_f$ and g, for ease of notation from now on we drop the parenthesis.

The vector field ξ can be mapped to $\mathscr{A}^\star\mathscr{D}_x^\star$,

$$\begin{aligned} \xi \mapsto X_\xi^\star &= X_{\xi\rho}^\star \partial_\rho^\star \\ X_\xi^\star \star f &= (X_{\xi\rho}^\star \partial_\rho^\star) \star f = X_{\xi\rho}^\star \star \partial_\rho f = \xi \cdot f. \end{aligned}$$

From the associativity in the algebra it follows

$$(X_\eta^\star \star X_\xi^\star) \star f = X_\eta^\star \star (X_\xi^\star \star f) = X_\eta^\star \star (\xi f) = \eta\xi f, \tag{1.44}$$

and therefore

$$X_\eta^\star \star X_\xi^\star - X_\xi^\star \star X_\eta^\star = X_{\eta\times\xi}^\star. \tag{1.45}$$

The deformed vector fields under the deformed Lie bracket form the same algebra as the vector fields under the ordinary Lie bracket.

1.6 Gauge transformations

Ordinary infinitesimal gauge transformations[7] are Lie algebra valued

$$\begin{aligned} \alpha(x) &= \alpha^a(x) T^a, \\ [T^a, T^b] &= i f^{abc} T^c. \end{aligned} \tag{1.46}$$

The action on a field is

$$\delta_\alpha \psi = i\alpha\psi = i\alpha^a(x) T^a \psi. \tag{1.47}$$

This can be reproduced by a star action on the field (cf. (1.36) and (1.38)):

$$\delta_\alpha^\star \psi = i X_{\alpha^a(x)}^\star \star T^a \psi = i\alpha \cdot \psi, \tag{1.48}$$

and represents the algebra via the commutator:

$$[\delta_\alpha^\star, \delta_\beta^\star] = \delta_\alpha^\star \delta_\beta^\star - \delta_\beta^\star \delta_\alpha^\star = -i\delta_{[\alpha,\beta]}^\star. \tag{1.49}$$

Gauge transformations of this kind have been introduced in [18–20]. Interesting is the transformation law of products of fields.

In the *undeformed* case we start from the transformation properties of the individual fields and transform the product as follows:

$$\begin{aligned} \delta_\alpha(\psi\chi) &= (\delta_\alpha\psi)\chi + \psi(\delta_\alpha\chi) \\ &= i\alpha^a\Big((T^a\psi)\chi + \psi(T^a\chi)\Big). \end{aligned} \tag{1.50}$$

[7] In this book ordinary, usual, or undeformed gauge transformations (gauge theory) refer to gauge transformations (gauge theory) on commutative spacetime.

In accordance with (1.48) we translate this to a star action

$$\delta^\star_\alpha(\psi\star\chi) = iX^\star_{\alpha^a}\star\{(T^a\psi)\star\chi+\psi\star(T^a\chi)\}\,. \tag{1.51}$$

The transformation law (1.51) is compatible with the algebra (1.49).

We now derive the deformed Leibniz rule obeyed by $\delta^\star_\alpha$. We rewrite (1.51) as

$$\delta^\star_\alpha(\psi\star\chi) = i\alpha^a\cdot\{(T^a\psi)\star\chi+\psi\star(T^a\chi)\}\,. \tag{1.52}$$

Expanding the right-hand side of (1.52) to first order in θ we obtain

$$\begin{aligned}\delta^\star_\alpha(\psi\star\chi) = i\alpha^a\Big\{ & T^a\psi\cdot\chi+\psi\cdot T^a\chi \\ & +\frac{ih}{2}\theta^{\rho\sigma}(T^a(\partial_\rho\psi)\cdot(\partial_\sigma\chi)+(\partial_\rho\psi)\cdot T^a(\partial_\sigma\chi))+\mathscr{O}(\theta^2)\Big\}.\end{aligned} \tag{1.53}$$

To compare this with the undeformed Leibniz rule $(\delta^\star_\alpha\psi)\star\chi+\psi\star(\delta^\star_\alpha\chi)$ we rewrite (1.53) by introducing the star product again and separating the terms that are of the form $\delta^\star_\alpha\psi = i\alpha\psi$ and $\delta^\star_\alpha\chi = i\alpha\chi$,

$$\begin{aligned}\delta^\star_\alpha(\psi\star\chi) &= (i\alpha\psi)\star\chi+\psi\star(i\alpha\chi) \\ & -\frac{ih}{2}\theta^{\rho\sigma}((i\partial_\rho\alpha^a)T^a\psi(\partial_\sigma\chi)+(\partial_\rho\psi)(i\partial_\sigma\alpha^a)T^a\chi)+\mathscr{O}(\theta^2).\end{aligned} \tag{1.54}$$

This expression can be extended to all orders in θ by induction.[8] The result is

[8] Let us redo the previous calculation in the second order of the deformation parameter. The proof by induction can then easily be derived by the same method.

First we expand the $\star$-product on the right-hand side of (1.52) to second order in the deformation parameter $\theta^{\rho\sigma}$

$$\begin{aligned}\delta^\star_\alpha(\psi\star\chi) = i\alpha^a\Big\{ & T^a\psi\cdot\chi+\psi\cdot T^a\chi \\ & +\frac{ih}{2}\theta^{\rho\sigma}\big(T^a(\partial_\rho\psi)\cdot(\partial_\sigma\chi)+(\partial_\rho\psi)\cdot T^a(\partial_\sigma\chi)\big) \\ & -\frac{h^2}{8}\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}\big(T^a(\partial_{\rho_1}\partial_{\rho_2}\psi)\cdot(\partial_{\sigma_1}\partial_{\sigma_2}\chi)+(\partial_{\rho_1}\partial_{\rho_2}\psi)\cdot T^a(\partial_{\sigma_1}\partial_{\sigma_2}\chi)\big)+\mathscr{O}(\theta^3)\Big\}.\end{aligned} \tag{1.55}$$

In the next step the terms of the form $\delta^\star_\alpha\psi$, $\delta^\star_\alpha\chi$, and similar are collected and the $\star$-product is reintroduced:

$$\begin{aligned}\delta^\star_\alpha(\psi\star\chi) &= i(\alpha^aT^a\psi)\star\chi+i\psi\star(\alpha^aT^a\chi) \\ & +i\left(-\frac{ih}{2}\theta^{\rho\sigma}\right)\Big([(\partial_\rho\alpha^a)T^a\psi]\star(\partial_\sigma\chi)+(\partial_\rho\psi)\star[(\partial_\sigma\alpha^a)T^a\chi]\Big) \\ & +i\frac{1}{2!}\left(-\frac{ih}{2}\right)^2\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}\Big([(\partial_{\rho_1}\partial_{\rho_2}\alpha^a)T^a\psi]\star(\partial_{\sigma_1}\partial_{\sigma_2}\chi) \\ & \qquad +(\partial_{\rho_1}\partial_{\rho_2}\psi)\star[(\partial_{\sigma_1}\partial_{\sigma_2}\alpha^a)T^a\chi]\Big)+\mathscr{O}(\theta^3).\end{aligned} \tag{1.56}$$

$$\begin{aligned}\delta_\alpha^\star(\psi\star\chi) &= i(\alpha^a T^a\psi)\star\chi + i\psi\star(\alpha^a T^a\chi)\\ &\quad + i\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{ih}{2}\right)^n \theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\{(\partial_{\rho_1}\dots\partial_{\rho_n}\alpha)\psi\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\chi)\\ &\quad + (\partial_{\rho_1}\dots\partial_{\rho_n}\psi)\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\alpha)\chi\}. \end{aligned} \tag{1.57}$$

The transformation law of the product of fields follows from the transformation law of the tensor product of fields. This is nicely expressed using the notion of coproduct, a main ingredient in the definition of a Hopf algebra [15]. Hopf algebras will be introduced and discussed in more detail in Chaps. 7 and 8. For *undeformed* gauge transformations we can write (1.50) in the Hopf algebra language

$$\delta_\alpha(\psi\otimes\chi) = i\Delta(\alpha)\psi\otimes\chi. \tag{1.58}$$

The coproduct $\Delta(\alpha)$ represents the Lie algebra in the tensor product representation

$$\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha, \tag{1.59}$$
$$[\Delta(\alpha),\Delta(\beta)] = \Delta([\alpha,\beta]). \tag{1.60}$$

The transformation law of the pointwise product can be defined with the multiplication μ:

$$\delta_\alpha(\psi\chi) = \mu\{\Delta(\alpha)\psi\otimes\chi\}. \tag{1.61}$$

The transformation law of the $\star$-product can be defined with the $\star$-multiplication $\mu_\star$ and the twisted coproduct. We define the twisted coproduct[9]

$$\Delta_{\mathscr{F}}(\alpha) = \mathscr{F}(\alpha\otimes 1 + 1\otimes\alpha)\mathscr{F}^{-1} \tag{1.62}$$

with

$$\mathscr{F} = e^{-\frac{ih}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}. \tag{1.63}$$

This twist $\mathscr{F}$ has all the properties that are required to define a Hopf algebra structure [21, 22]. We can show that

$$\delta_\alpha^\star(\psi\star\chi) = \mu_\star\{\Delta_{\mathscr{F}}(\alpha)\psi\otimes\chi\} \tag{1.64}$$

by a direct calculation performed expanding order by order in θ.[10]

Note that the term $i(\partial_\rho\alpha^a)T^a\psi$ can be interpreted as the transformation law of the field ψ with the gauge parameter $\partial_\rho\alpha$ that is $\delta^\star_{\partial_\rho\alpha}\psi$. Equation (1.56) gives the deformed Leibniz rule to second order in the deformation parameter $\theta^{\rho\sigma}$. Repeating this calculation for higher orders leads to the deformed Leibniz rule (1.57).

[9] The term twisted coproduct emphasizes that $\Delta_{\mathscr{F}}$ is a deformation of the undeformed coproduct (1.60). $\Delta_{\mathscr{F}}$ has all the properties of a Hopf algebra coproduct. Similarly in this book by twisted Hopf algebra we mean a Hopf algebra that is obtained deforming (via a twist $\mathscr{F}$) another Hopf algebra, typically the Hopf algebra associated with a Lie algebra.

[10] For example, at first order in θ we have

$$\Delta_{\mathscr{F}}(\alpha) = \mathscr{F}(\alpha\otimes 1+1\otimes\alpha)\mathscr{F}^{-1} = \alpha\otimes 1 + 1\otimes\alpha - \frac{ih}{2}\theta^{\rho\sigma}[\partial_\rho\otimes\partial_\sigma, \alpha\otimes 1 + 1\otimes\alpha] + \mathscr{O}(\theta^2)$$

1.7 Diffeomorphism

Infinitesimal diffeomorphisms are vector fields. They are elements of $\mathscr{A}\mathscr{D}_x$

$$\xi = \xi^\mu(x)\partial_\mu. \tag{1.65}$$

Their product in $\mathscr{A}\mathscr{D}_x$ is

$$\xi\eta = \xi^\mu(x)\big(\partial_\mu\eta^\rho(x)\big)\partial_\rho + \xi^\mu(x)\eta^\rho(x)\partial_\mu\partial_\rho. \tag{1.66}$$

Through the Lie bracket we obtain the Lie algebra of diffeomorphism

$$\begin{aligned}[\xi,\eta] &= \Big(\xi^\mu(\partial_\mu\eta^\rho) - \eta^\mu(\partial_\mu\xi^\rho)\Big)\partial_\rho \\ &= (\xi\times\eta)^\rho\partial_\rho = \xi\times\eta.\end{aligned} \tag{1.67}$$

The vector field ξ, an element of $\mathscr{A}\mathscr{D}_x$, can be mapped to $\mathscr{A}^\star\mathscr{D}^\star_x$

$$\begin{aligned}\mathscr{A}\mathscr{D}_x &\to \mathscr{A}^\star\mathscr{D}^\star_x \\ \xi &\mapsto X^\star_\xi = X^\star_{\xi\rho}\partial^\star_\rho.\end{aligned} \tag{1.68}$$

When it $\star$-acts on a function $f \in \mathscr{A}_x$ we obtain

$$X^\star_\xi \star f = (X^\star_{\xi\rho}\partial^\star_\rho)\star f = X^\star_{\xi\rho}\star\partial_\rho f = \xi\cdot f. \tag{1.69}$$

This is analogous to (1.38) and we can proceed as there. From associativity in $\mathscr{A}\mathscr{D}_x$ and $\mathscr{A}^\star\mathscr{D}^\star_x$ follows

$$(X^\star_\eta \star X^\star_\xi)\star f = X^\star_\eta \star (X^\star_\xi \star f) = X^\star_\eta \star (\xi f) = \eta\xi f, \tag{1.70}$$

and therefore

$$X^\star_\eta \star X^\star_\xi - X^\star_\xi \star X^\star_\eta = X^\star_{\eta\times\xi}. \tag{1.71}$$

The deformed vector fields under the Lie bracket $[X^\star_\eta \stackrel{\star}{,} X^\star_\chi] = X^\star_\eta \star X^\star_\xi - X^\star_\xi \star X^\star_\eta$ close the same algebra as the vector fields under the ordinary Lie bracket. They represent the deformed algebra[11] of diffeomorphisms.

Based on this deformed algebra of diffeomorphisms the Einstein theory of gravity on noncommutative (deformed) space has been constructed in [21, 22]. The coproduct of the diffeomorphisms algebra has to be modified as before for gauge theories. It is the first theory of gravity defined on a deformed space and is under investigation now.

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and recalling (1.33) we obtain $\Delta_{\mathscr{F}}(\alpha) = \alpha\otimes 1 + 1\otimes\alpha - \frac{ih}{2}\theta^{\rho\sigma}((\partial_\rho\alpha)\otimes\partial_\sigma + \partial_\rho\otimes(\partial_\sigma\alpha))$. This agrees with (1.54).

[11] Deformed or noncommutative algebra of diffeomorphisms refers to the algebra of diffeomorphisms on the noncommutative space.

1.8 Conclusion

The formalism developed here opens a way to construct a deformation of differential geometry and therefore deformed gauge theories and gravity theories. Mathematically it is certainly an interesting possibility. If physics knows anything about it is hard to say. Future investigation might shed some light on this question. I am left to quote Riemann and express my hope that

> ...this work is not hindered by too restricted concepts and that the progress in comprehending the connection of things is not obstructed by traditional prejudices.

1.9 Appendix

In this appendix[12] we discuss in more detail the application of the Poincaré–Birkhoff–Witt property and derive explicitly the Moyal product (1.22).

Consider the algebra $\mathscr{P}_x$ of polynomial functions in N commuting coordinates $x^1, \dots, x^N$. Any function can be expanded in the monomial basis

$$\begin{aligned} f(x) &= \sum_j C_{\mu_1 \dots \mu_j} x^{\mu_1} \dots x^{\mu_j} \\ &= C + C_\mu x^\mu + C_{\mu\nu} x^\mu x^\nu + \cdots \end{aligned} \tag{1.72}$$

and is uniquely determined by the coefficients $C_{\mu_1 \dots \mu_j}$ that are completely symmetric in their indices.

Consider an algebra $\hat{\mathscr{P}}_{\hat{x}}$ of polynomial functions in N noncommuting coordinates $\hat{x}^1, \dots, \hat{x}^N$ and with the PBW property. The PBW property enables the introduction of a basis of ordered monomials. There are many possible orderings. The most often used ones are the symmetric and the normal ordering. If we chose the symmetric ordering (we denote the ordering by : :), the basis in the algebra is given by

$$\begin{aligned} :1: &= 1, \\ :\hat{x}^\mu: &= \hat{x}^\mu, \\ :\hat{x}^\mu \hat{x}^\nu: &= \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu), \\ \dots&. \end{aligned} \tag{1.73}$$

An arbitrary element of $\hat{\mathscr{P}}_{\hat{x}}$ is then written as an expansion in the basis (1.73)

$$\begin{aligned} \hat{f}(\hat{x}) &= \sum_j C_{\mu_1 \dots \mu_j} :\hat{x}^{\mu_1} \dots \hat{x}^{\mu_j}: \\ &= C + C_\mu :\hat{x}^\mu: + C_{\mu\nu} :\hat{x}^\mu \hat{x}^\nu: + \cdots, \end{aligned} \tag{1.74}$$

[12] This appendix, the footnotes in Chaps. 1–3, and the note at the end of Sect. 3.3 have been added by Paolo Aschieri and Marija Dimitrijević.

and it is fully characterized by the completely symmetric coefficients $C_{\mu_1 \ldots \mu_j}$.

We call W[13] the isomorphism between the vector spaces $\mathscr{P}_x$ and $\hat{\mathscr{P}}_{\hat{x}}$ obtained by mapping the basis of $\mathscr{P}_x$ into the basis of $\hat{\mathscr{P}}_{\hat{x}}$ selected by the chosen ordering prescription. Explicitly,

$$\begin{array}{rl} f(x) & = C + C_\mu x^\mu \;+\; C_{\mu\nu} x^\mu x^\nu + \cdots \\ W \Big\downarrow & \\ \hat{f}(\hat{x}) & = C + C_\mu : \hat{x}^\mu : + C_{\mu\nu} : \hat{x}^\mu \hat{x}^\nu : + \cdots . \end{array} \tag{1.75}$$

In the case of symmetric ordering, an equivalent expression (see also Sect. 6.3) is given by

$$\hat{f}(\hat{x}) = W(f) = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N k \; \tilde{f}(k) e^{ik_\rho \hat{x}^\rho} , \tag{1.76}$$

where $\tilde{f}(k)$ is the usual Fourier transform of $f(x)$

$$\tilde{f}(k) = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N x \; f(x) e^{-ik_\rho x^\rho} . \tag{1.77}$$

Indeed $\hat{f}(\hat{x})$ is a sum of fully symmetrized monomials because for any value of k_ρ and any power n, the expression $(k_\rho \hat{x}^\rho)^n$ is fully symmetrized.

For an arbitrary monomial we have

$$\begin{aligned} W(x^{\mu_1} \ldots x^{\mu_j}) &= (i)^j \int \mathrm{d}^N k \, \left(\partial_{k_{\mu_1}} \ldots \partial_{k_{\mu_j}} \delta^{(N)}(k) \right) e^{ik_\rho \hat{x}^\rho} \\ &= (i)^j (-1)^j \int \mathrm{d}^N k \; \delta^{(N)}(k) \left(\partial_{k_{\mu_1}} \ldots \partial_{k_{\mu_j}} e^{ik_\rho \hat{x}^\rho} \right) \\ &= (i)^{2j} (-1)^j \frac{1}{j!} \sum_{\sigma \in S_j} (\hat{x}^{\sigma(\mu_1)} \ldots \hat{x}^{\sigma(\mu_j)}) \\ &= : \hat{x}^{\mu_1} \ldots \hat{x}^{\mu_j} : . \end{aligned} \tag{1.78}$$

The $\star$-product is defined by

$$W(f \star g) = W(f) \cdot W(g) = \hat{f}(\hat{x}) \cdot \hat{g}(\hat{x}). \tag{1.79}$$

Let us derive this $\star$-product in the case of θ-deformed space, defined by relations (1.11). We start from

$$W(f) \cdot W(g) = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N k \; \tilde{f}(k) e^{ik_\rho \hat{x}^\rho} \cdot \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N p \; \tilde{g}(p) e^{ip_\rho \hat{x}^\rho}$$

[13] The symbol W refers to Weyl since he was the first one to introduce this procedure in quantum mechanics [16, 17].

$$= \frac{1}{(2\pi)^N} \int \mathrm{d}^N k \int \mathrm{d}^N p \; \tilde{f}(k)\tilde{g}(p) e^{ik_\rho \hat{x}^\rho} e^{ip_\sigma \hat{x}^\sigma}.$$

Since the exponents do not commute (coordinates $\hat{x}^\mu$ do not commute) one has to use the Campbell–Baker–Hausdorff (CBH) formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+\cdots}, \tag{1.80}$$

where A and B are two noncommuting operators. In the case of θ-deformed space the CBH formula terminates, terms with more than one commutator vanish, and we obtain

$$\begin{aligned} W(f)\cdot W(g) &= \frac{1}{(2\pi)^N} \int \mathrm{d}^N k \int \mathrm{d}^N p \; \tilde{f}(k)\tilde{g}(p) e^{i(k+p)_\rho \hat{x}^\rho - \frac{i}{2}\theta^{\rho\sigma}k_\rho p_\sigma} \qquad (1.81)\\ &= \frac{1}{(2\pi)^N} \int \mathrm{d}^N k \int \mathrm{d}^N q \; \tilde{f}(k)\tilde{g}(q-k) e^{iq_\rho \hat{x}^\rho - \frac{i}{2}\theta^{\rho\sigma}k_\rho (q-k)_\sigma}. \end{aligned}$$

In the last line a change of variables $(k+p)^\rho = q^\rho$ is performed. Comparing this expression with

$$W(f\star g) = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N q \; \widetilde{f\star g}(q)\, e^{iq_\rho \hat{x}^\rho}, \tag{1.82}$$

we conclude

$$\widetilde{f\star g}(q) = \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N k \; \tilde{f}(k)\tilde{g}(q-k) e^{-\frac{i}{2}\theta^{\rho\sigma}k_\rho (q-k)_\sigma}. \tag{1.83}$$

The last step is the inverse Fourier transform

$$\begin{aligned} f\star g(x) &= \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N q \left(\frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N k \; \tilde{f}(k)\tilde{g}(q-k) e^{-\frac{i}{2}\theta^{\rho\sigma}k_\rho (q-k)_\sigma} \right) e^{-iq_\sigma x^\sigma}\\ &= \frac{1}{(2\pi)^N} \int \mathrm{d}^N p \int \mathrm{d}^N k \; \tilde{f}(k) e^{-ik_\sigma x^\sigma} e^{-\frac{i}{2}k_\rho \theta^{\rho\sigma} p_\sigma} \tilde{g}(p) e^{-ip_\sigma x^\sigma}. \end{aligned}$$

In the last line the change of variables $(q-k)_\mu = p_\mu$ is performed. In order to evaluate this integral we expand in powers of the deformation parameter $\theta^{\rho\sigma}$, calculate term by term, and then sum up all the terms again

$$\begin{aligned} f\star g(x) &= \frac{1}{(2\pi)^N} \int \mathrm{d}^N q \int \mathrm{d}^N k \; \tilde{f}(k) e^{-ik_\sigma x^\sigma} \Big(1 - \frac{i}{2} k_\rho \theta^{\rho\sigma} p_\sigma \\ &\quad + \frac{1}{2!}\left(-\frac{i}{2}\right)^2 k_{\rho_1} k_{\rho_2} \theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2} p_{\sigma_1} p_{\sigma_2} + \cdots \Big) \tilde{g}(p) e^{-ip_\sigma x^\sigma}\\ &= fg + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g) - \frac{1}{8}\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}(\partial_{\rho_1}\partial_{\rho_2} f)(\partial_{\sigma_1}\partial_{\sigma_2} g) + \cdots\\ &= \mu\{ e^{\frac{i}{2}h\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g \}. \end{aligned} \tag{1.84}$$

The pointwise multiplication was defined in (1.23).

For polynomial functions expression (1.84) is a finite sum and therefore we have a well-defined $\star$-product. On the other hand, in general for f and g smooth functions we have an infinite sum that not always converges. One route is to consider different expressions for the $\star$-product that however reduce to the above one for polynomial functions. These expressions are typically via an integral kernel and therefore are nonlocal, an example is

$$f \star g\,(x) = (2\pi)^{-2N} \int\!\!\int f\left(x + \frac{1}{2}\theta u\right) g(x+s) e^{ius} \mathrm{d}^N u\, \mathrm{d}^N s\ .$$

This product is well defined on the space of smooth rapidly decreasing functions.

A different route is to continue to work with a $\star$-product that is a differential operator in both its arguments (i.e., a bidifferential operator). This is achieved by introducing the formal parameter h and by considering the algebra $\mathscr{A}_x$ of formal power series in h of smooth functions. The $\star$-product (1.84) is well defined on $\mathscr{A}_x$ and we obtain the deformed algebra $\mathscr{A}_x^\star$.

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Chapter 2
Deformed Gauge Theories

Julius Wess

Gauge theories are studied on a space of functions with the Moyal product. The development of these ideas follows the differential geometry of the usual gauge theories, but several changes are forced upon us. The Leibniz rule has to be changed such that the theory is now based on a twisted Hopf algebra. Nevertheless, this twisted symmetry structure leads to conservation laws. The symmetry has to be extended from Lie algebra valued to enveloping algebra valued and new vector potentials have to be introduced. As usual, field equations are subjected to consistency conditions that restrict the possible models. Some examples are studied.

2.1 Introduction

Gauge theories have been formulated and developed on the algebra of functions with a pointwise product:

$$\mu\{f \otimes g\} = f \cdot g. \tag{2.1}$$

This product is associative and commutative.

Recently, algebras of functions with a deformed product have been studied intensively [1–5]. These deformed (star) products remain associative but not commutative.

The simplest example is the Moyal product,[1] see Chap. 1 for details

$$\mu_\star\{f \otimes g\} = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\}. \tag{2.2}$$

It had its first appearance in quantum mechanics [6, 7].

The star product can be seen as a higher order f-dependent differential operator acting on the function g. For the example of the Moyal product this is

[1] Note that in this and in the following chapters in the first part of the book the deformation parameter h is absorbed in $\theta^{\rho\sigma}$. Therefore, from now on we refer to $\theta^{\rho\sigma}$ as the deformation parameter.

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$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{\rho_1 \sigma_1} \ldots \theta^{\rho_n \sigma_n} \left(\partial_{\rho_1} \ldots \partial_{\rho_n} f\right) \left(\partial_{\sigma_1} \ldots \partial_{\sigma_n} g\right). \tag{2.3}$$

The differential operator maps the function g to the function $f \star g$.

The inverse map also exists [8, 9]. It $\star$-maps the function g to the function obtained by pointwise multiplying it with f

$$X_f^\star \star g = f \cdot g \tag{2.4}$$

For the Moyal product we obtain

$$X_f^\star = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2}\right)^n \theta^{\rho_1 \sigma_1} \ldots \theta^{\rho_n \sigma_n} \left(\partial_{\rho_1} \ldots \partial_{\rho_n} f\right) \star \partial^\star_{\sigma_1} \ldots \partial^\star_{\sigma_n}. \tag{2.5}$$

The star-acting derivatives we denote by $\partial^\star_\rho$. For the Moyal product the $\star$-derivatives and the usual derivatives are the same. Star differentiation and star differential operators have been thoroughly discussed in Chap. 1 and in [9, 10].

In this chapter we are going to study gauge transformations on Moyal or θ-deformed spaces.[2]

2.2 Gauge transformations

Undeformed infinitesimal gauge transformations are Lie algebra valued:

$$\begin{aligned}
&\delta_\alpha \phi(x) = i\alpha(x)\phi(x), \\
&\alpha(x) = \sum_a \alpha^a(x) T^a, \\
&[T^a, T^b] = i f^{abc} T^c, \\
&[\delta_\alpha, \delta_\beta]\phi = [\alpha, \beta]\phi = -i\delta_{[\alpha,\beta]}\phi,
\end{aligned} \tag{2.6}$$

where $\phi(x)$ is a matter field which belongs to an irreducible representation of the gauge group.

In the previous chapter *deformed* gauge transformations were introduced. Here we analyze them in more detail. They are defined as follows [11, 12]:

$$\delta^\star_\alpha \phi = i X^\star_\alpha \star \phi = i X^\star_{\alpha^a} T^a \star \phi = i\alpha \cdot \phi. \tag{2.7}$$

From the fact that $X_f^\star \star X_g^\star = X^\star_{f \cdot g}$, we conclude

$$\begin{aligned}
[X^\star_\alpha \stackrel{\star}{,} X^\star_\beta] &= X^\star_{-i[\alpha,\beta]}, \\
[\delta^\star_\alpha, \delta^\star_\beta]\phi &= -i\delta^\star_{[\alpha,\beta]}\phi.
\end{aligned} \tag{2.8}$$

[2] A comparison between the present approach to noncommutative gauge theories and an earlier one, so-called Seiberg–Witten map approach, is in Chap. 5.

The $\star$-transformations $\delta^\star_\alpha$ represent the algebra via the usual[3] commutator. However, written in terms of the operators $X^\star_\alpha$ the same algebra is represented via the $\star$-commutator.

Before we construct gauge theories we have to learn how products of fields transform.

In the *undeformed* situation we use, without even thinking, the Leibniz rule:

$$\delta_\alpha(\phi\cdot\psi) = (\delta_\alpha\phi)\cdot\psi + \phi\cdot(\delta_\alpha\psi), \tag{2.9}$$

and we can easily verify that this Leibniz rule is consistent with the Lie algebra:

$$[\delta_\alpha,\delta_\beta](\phi\cdot\psi) = -i\delta_{[\alpha,\beta]}(\phi\cdot\psi). \tag{2.10}$$

For the *deformed* transformation law of a $\star$-product of fields we demand a transformation law that is in the class of transformations defined in (2.7) [8, 9, 11, 13, 14]. This amounts to first decomposing the representation $\phi\star\psi$ for x-independent parameters into its irreducible parts and then follow (2.7) for gauging

$$\delta^\star_\alpha(\phi\star\psi) = iX^\star_{\alpha^a}\star\{T^a\phi\star\psi + \phi\star T^a\psi\}. \tag{2.11}$$

Certainly it is consistent with the Lie algebra:

$$[\delta^\star_\alpha,\delta^\star_\beta](\phi\star\psi) = -i\delta^\star_{[\alpha,\beta]}(\phi\star\psi). \tag{2.12}$$

Because $\phi\star\psi$ is a function we can use the definition of $X^\star_f$ given in (2.4) and simplify (2.11)

$$\delta^\star_\alpha(\phi\star\psi) = i\alpha^a\cdot\{T^a\phi\star\psi + \phi\star T^a\psi\}. \tag{2.13}$$

As α^a does not commute with the $\star$-operation this is different from (2.9). To see this difference more clearly we expand (2.13) in θ

$$\begin{aligned}\delta^\star_\alpha(\phi\star\psi) = i\alpha^a\Big\{&T^a\phi\cdot\psi + \phi\cdot T^a\psi\\ &+\frac{i}{2}\theta^{\rho\sigma}\left(T^a\partial_\rho\phi\cdot\partial_\sigma\psi + \partial_\rho\phi\cdot T^a\partial_\sigma\psi\right) + O(\theta^2)\Big\}.\end{aligned} \tag{2.14}$$

The final version of the Leibniz rule for the $\star$-product should be entirely expressed with $\star$-operations. Thus we express (2.14) with $\star$-products. A short calculation (see Chap. 1, Sect. 1.6 for details) shows

$$\begin{aligned}\delta^\star_\alpha(\phi\star\psi) &= i(\alpha\phi)\star\psi + i\phi\star(\alpha\psi)\\ &\quad-\frac{i}{2}\theta^{\rho\sigma}\left(i\left((\partial_\rho\alpha)\phi\right)\star(\partial_\sigma\psi) + (\partial_\rho\phi)\star i\left((\partial_\sigma\alpha)\psi\right)\right) + O(\theta^2).\end{aligned} \tag{2.15}$$

[3] Here the usual commutator $[A,B] = AB - BA$ stands in contrast to the $\star$-commutator which is defined in the following way $[A \stackrel{\star}{,} B] = A\star B - B\star A$.

With more work we can prove by induction to all orders in θ the following equation:

$$\begin{aligned}\delta_\alpha^\star(\phi\star\psi) &= i(\alpha\phi)\star\psi + i\phi\star(\alpha\psi)\\ &\quad + i\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\right)^n \theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\{(\partial_{\rho_1}\dots\partial_{\rho_n}\alpha)\phi\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)\\ &\quad + (\partial_{\rho_1}\dots\partial_{\rho_n}\phi)\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\alpha)\psi\}. \end{aligned} \tag{2.16}$$

This is different from what we obtain by putting just stars in the Leibniz rule (2.9). But this difference has a well-defined meaning if we use the Hopf algebra language to derive the Leibniz rule.

2.3 Hopf algebra techniques

The essential ingredient for a Hopf algebra [15, 16] is the comultiplication $\Delta(\alpha)$: For the *undeformed* situation we define

$$\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha. \tag{2.17}$$

It allows us to write the Leibniz rule (2.9) in the Hopf algebra language:

$$\delta_\alpha(\phi\cdot\psi) = \mu\{\Delta(\alpha)\phi\otimes\psi\}. \tag{2.18}$$

In the *deformed* situation we use a twisted coproduct:

$$\begin{aligned}\Delta_{\mathscr{F}}(\alpha) &= \mathscr{F}(\alpha\otimes 1 + 1\otimes\alpha)\mathscr{F}^{-1},\\ \mathscr{F} &= e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}. \end{aligned} \tag{2.19}$$

Here $\mathscr{F}$ is a twist that has all the properties to define a Hopf algebra with $\Delta_{\mathscr{F}}(\alpha)$ as a comultiplication [17–24]. Details about Hopf algebra methods, twists, and twisted Hopf algebras will be given in Chaps. 7 and 8. We can show that the transformation (2.16) can be written in the form

$$\delta_\alpha^\star(\phi\star\psi) = i\mu_\star\{\Delta_{\mathscr{F}}(\alpha)\phi\otimes\psi\}, \tag{2.20}$$

with the multiplication $\mu_\star$ defined in (2.2). Equation (2.20) defines the Leibniz rule in terms of the twisted comultiplication and the product $\mu_\star$. To show this we start from Eq. (2.13) and write it with the explicit definition of the $\star$-product:

$$\begin{aligned}\delta_\alpha^\star(\phi\star\psi) &= i\alpha^a\mu\left\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}(T^a\phi\otimes\psi + \phi\otimes T^a\psi)\right\}\\ &= i\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\left(\alpha^a T^a(\partial_{\rho_1}\dots\partial_{\rho_n}\phi)(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)\right.\\ &\quad \left.+(\partial_{\rho_1}\dots\partial_{\rho_n}\phi)\alpha^a T^a(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)\right). \end{aligned} \tag{2.21}$$

This we now rewrite as follows:

$$\begin{aligned}\delta_\alpha^\star(\phi\star\psi) &= i\mu(\alpha\otimes 1+1\otimes\alpha)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\phi\otimes\psi\\ &= i\mu\left\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\cdot e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}(\alpha\otimes 1+1\otimes\alpha)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\phi\otimes\psi\right\}\\ &= i\mu_\star\{\Delta_{\mathscr{F}}(\alpha)\phi\otimes\psi\}.\end{aligned} \tag{2.22}$$

The last line is exactly (2.20).

Gauge fields can be included in this formalism as well. In the *undeformed* situation they are Lie algebra valued, $A_\mu(x)=A_\mu^a(x)T^a$, and under infinitesimal gauge transformations transform as follows:

$$\delta A_\mu=\partial_\mu\alpha+i\alpha^a[T^a,A_\mu]. \tag{2.23}$$

Let us calculate the contribution of the gauge field to the Leibniz rule. As an example we calculate

$$\delta_\alpha^\star(A_\mu\star\phi)=\mu_\star\{\Delta_{\mathscr{F}}(\alpha)A_\mu\otimes\phi\} \tag{2.24}$$

and obtain

$$\begin{aligned}\delta_\alpha^\star(A_\mu\star\psi) &= i\alpha^a\left([T^a,A_\mu]\star\psi\right)+i\alpha^a\left(A_\mu\star T^a\psi\right)+(\partial_\mu\alpha^a)T^a\psi\\ &= i\alpha^a\left((T^aA_\mu)\star\psi-(A_\mu T^a)\star\psi\right)\\ &\quad +i\alpha^a(A_\mu T^a)\star\psi+(\partial_\mu\alpha^a)T^a\psi\\ &= i\alpha^aT^a(A_\mu\star\psi)+(\partial_\mu\alpha)\psi.\end{aligned} \tag{2.25}$$

Now we define a covariant derivative

$$D_\mu^\star\psi=\partial_\mu\psi-iA_\mu\star\psi. \tag{2.26}$$

It will transform covariantly

$$\delta_\alpha^\star(D_\mu^\star\psi)=i\alpha^aT^a(D_\mu^\star\psi)=iX_{\alpha^a}^\star\star T^a(D_\mu^\star\psi), \tag{2.27}$$

if the vector field A_μ transforms as in (2.23)

$$\delta_\alpha^\star A_\mu=\partial_\mu\alpha+i\alpha^a[T^a,A_\mu]=\partial_\mu\alpha+iX_{\alpha^a}^\star\star[T^a,A_\mu]. \tag{2.28}$$

From (2.28) we see that a Lie algebra valued vector field remains Lie algebra valued by the transformation (2.28).

2.4 Field equations

Now we proceed as in the *undeformed* situation. First we define the field strength tensor:

$$F_{\mu\nu}=i[D_\mu^\star \stackrel{\star}{,} D_\nu^\star]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu]. \tag{2.29}$$

Here we see already that $F_{\mu\nu}$ will not be Lie algebra valued even for Lie algebra-valued vector fields. Namely, assuming that the gauge field is Lie algebra valued $A_\mu = A^a_\mu T^a$ the field strength tensor $F_{\mu\nu}$ (2.29) can be decomposed in two parts

$$F_{\mu\nu} = F^a_{1\mu\nu} T^a + F^{ab}_{2\mu\nu} \frac{1}{2}\{T^a, T^b\}. \tag{2.30}$$

Since anticommutator of generators $\{T^a, T^b\}$ is not Lie algebra valued in general, the full $F_{\mu\nu}$ will not be Lie algebra valued in general.

Using the twisted gauge transformations of the gauge field A_μ (2.28) and the deformed Leibniz rule (2.16) we derive the transformation law of the field strength tensor:

$$\delta^\star_\alpha F_{\mu\nu} = iX^\star_{\alpha^a} \star [T^a, F_{\mu\nu}] = i[\alpha, F_{\mu\nu}]. \tag{2.31}$$

The expression $F^{\mu\nu} \star F_{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$ will transform accordingly

$$\delta^\star_\alpha (F^{\mu\nu} \star F_{\mu\nu}) = iX^\star_{\alpha^a} \star [T^a, F^{\mu\nu} \star F_{\mu\nu}] = i[\alpha, F^{\mu\nu} \star F_{\mu\nu}]. \tag{2.32}$$

Hint, use the transformation law (2.31) and the deformed Leibniz rule (2.16).

The Lagrangian that is invariant under the twisted gauge transformations (2.28) we define as in the gauge theory on commutative space:

$$\mathscr{L} = \frac{1}{c} \mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}), \tag{2.33}$$

where c is an arbitrary constant. It is invariant and it is a deformation[4] of the *undeformed* Lagrangian of a gauge theory.

To speak about an action we have to define integration. We take the usual integral over x on the commutative space and we can verify that

$$\int \mathrm{d}^4x\, f \star g = \int \mathrm{d}^4x\, g \star f = \int \mathrm{d}^4x\, f \cdot g \tag{2.34}$$

by partial integration. This is called the trace property of the integral or cyclicity .

Equation (2.34) allows a cyclic permutation of the fields under the integral. To derive the field equations we use the usual Leibniz rule for the functional variation, that is, we vary the field where it stands. The trace property is then used to derive the final result. As an example we look at the action for the gauge field

$$S = \frac{1}{c} \int \mathrm{d}^4x\, \mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}). \tag{2.35}$$

[4] One can expand the $\star$-products appearing in the Lagrangian (2.33) and check that in the zeroth order in the deformation parameter $\theta^{\rho\sigma}$ the Lagrangian of the undeformed theory is obtained. Higher order terms give new contributions due to the noncommutativity (deformation) of the commutative space.

From the trace property we compute

$$\begin{aligned}
\frac{\delta S}{\delta A_\rho(z)} &= \frac{1}{c}\frac{\delta}{\delta A_\rho(z)}\int \mathrm{d}^4x\ \mathrm{Tr}(F^{\mu\nu}\star F_{\mu\nu}) \\
&= \frac{1}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\left(\left(\frac{\delta F^{\mu\nu}(x)}{\delta A_\rho(z)}\right)\star F_{\mu\nu} + F^{\mu\nu}\star\left(\frac{\delta F_{\mu\nu}(x)}{\delta A_\rho(z)}\right)\right) \\
&= \frac{2}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\frac{\delta F_{\mu\nu}(x)}{\delta A_\rho(z)}\star F^{\mu\nu}(x) \\
&= \frac{2}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\frac{\delta}{\delta A_\rho(z)}(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu])\star F^{\mu\nu}(x) \\
&= \frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\frac{\delta}{\delta A_\rho(z)}(\partial_\mu A_\nu - iA_\mu\star A_\nu)\star F^{\mu\nu}(x)
\end{aligned} \tag{2.36}$$

because $F^{\mu\nu}$ is antisymmetric. Then we have

$$\begin{aligned}
\frac{\delta S}{\delta A_\rho(z)} &= \frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\{-\delta^{(4)}(x-z)\star(\partial_\mu F^{\mu\rho}) \\
&\quad -i\delta^{(4)}(x-z)\star A_\mu\star F^{\rho\mu} - iA_\mu\star\delta^{(4)}(x-z)\star F^{\mu\rho}\} \\
&= -\frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\delta^{(4)}(x-z)\star\{\partial_\mu F^{\mu\rho} - iA_\mu\star F^{\mu\rho} + iF^{\mu\rho}\star A_\mu\}.
\end{aligned} \tag{2.37}$$

The field equations follow after using (2.34)

$$\frac{\delta S}{\delta A_\rho(z)} = -\frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\delta^{(4)}(x-z)\{\partial_\mu F^{\mu\rho} - iA_\mu\star F^{\mu\rho} + iF^{\mu\rho}\star A_\mu\}. \tag{2.38}$$

These are exactly the equations we have expected from covariance:

$$D^\star_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - i[A_\mu \stackrel{\star}{,} F^{\mu\nu}] = 0. \tag{2.39}$$

We have already seen that $F_{\mu\nu}$ cannot be Lie algebra valued. From the field equations (2.39), considered as equations for the vector potential A_μ, we see that A_μ cannot be Lie algebra valued either. We have to consider $F_{\mu\nu}$ and A_μ to be enveloping algebra valued. The additional vector fields (coming from the non-Lie algebra-valued parts) will introduce additional ghosts in the Lagrangian. To eliminate them we have to enlarge the symmetry to be enveloping algebra valued as well. For simplicity we assume α, A_μ, and $F_{\mu\nu}$ to be matrix valued when the matrices act in the representation space of T^a.

From the field equations (2.39) follows a consistency equation because $F^{\mu\nu}$ is antisymmetric in μ and ν:

$$\partial_\nu[A_\mu \stackrel{\star}{,} F^{\mu\nu}] = 0. \tag{2.40}$$

To verify this condition we have to use the field equations (2.39). First we differentiate (2.40)

$$\partial_\nu[A_\mu \stackrel{\star}{,} F^{\mu\nu}] = [\partial_\nu A_\mu \stackrel{\star}{,} F^{\mu\nu}] + [A_\mu \stackrel{\star}{,} \partial_\nu F^{\mu\nu}]. \tag{2.41}$$

In the first term we replace $\partial_\nu A_\mu$ by $\frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu)$ because $F_{\mu\nu}$ is antisymmetric in μ and ν. Then we express this term by $F_{\mu\nu}$ according to (2.29):

$$\frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu) = \frac{i}{2}F_{\nu\mu} + \frac{i}{2}[A_\nu \overset{\star}{,} A_\mu]. \tag{2.42}$$

The $\star$-commutator $[F^{\mu\nu} \overset{\star}{,} F_{\mu\nu}] = F^{\mu\nu} \star F_{\mu\nu} - F_{\mu\nu} \star F^{\mu\nu}$ vanishes and we are left with $\frac{i}{2}[[A_\nu \overset{\star}{,} A_\mu] \overset{\star}{,} F^{\mu\nu}]$ for the first term in (2.41). For the second term in (2.41) we use the field equations (2.39). Finally all terms left add up to zero if we use the Jacobi identity. In all these equations A_μ and $F_{\mu\nu}$ are supposed to be matrices. We have suppressed the matrix indices.

A conserved current is found

$$j^\nu = [A_\mu \overset{\star}{,} F^{\mu\nu}], \qquad \partial_\nu j^\nu = 0. \tag{2.43}$$

For $\theta^{\rho\sigma} = 0$ this is the current of a non-abelian gauge theory **on commutative space**.

2.5 Matter fields

Matter fields can be coupled covariantly to the gauge fields via a covariant derivative. We start from a multiplet of the gauge group ψ_A not necessarily irreducible. The index A denotes the component of the field ψ in the representation space. The transformation law of ψ is $\delta^\star_\alpha \psi_A = iX^\star_{\alpha_{AB}} \star \psi_B = i\alpha_{AB}\psi_B$. For the usual gauge transformations α_{AB} will be Lie algebra valued. The covariant derivative is

$$(D^\star_\mu \psi)_A = \partial_\mu \psi_A - iA_{\mu AB} \star \psi_B. \tag{2.44}$$

The gauge potential A_μ in now supposed to be matrix valued in the representation space spanned by the matter fields.

For a spinor field

$$\bar{\psi}_{\alpha A} \star \gamma^\mu_{\alpha\beta}(D^\star_\mu \psi)_A \tag{2.45}$$

will be invariant and therefore suitable for a covariant Lagrangian.

We consider the Lagrangian

$$\mathscr{L} = \frac{1}{c}\mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}) + \bar{\psi} \star \gamma^\mu (i\partial_\mu + A_\mu \star)\psi - m\bar{\psi} \star \psi. \tag{2.46}$$

We have suppressed the matrix indices.

The field equations are obtained from (2.46) by varying the fields in the same way as in Sect. 2.4:

$$\frac{\delta \mathscr{L}}{\delta A_\rho} = \partial_\mu F^{\mu\rho}_{AB} + i[A_\mu \overset{\star}{,} F^{\rho\mu}]_{AB} + \gamma^\rho_{\alpha\beta}\psi_{\beta A} \star \bar{\psi}_{\alpha B} = 0, \tag{2.47}$$

and for the matter fields

$$\frac{\delta \mathscr{L}}{\delta \bar{\psi}} = \gamma^\mu (\partial_\mu \psi_A - i A_{\mu AB} \star \psi_B) + im\psi_A = 0 \tag{2.48}$$
$$\frac{\delta \mathscr{L}}{\delta \psi} = (\partial_\mu \bar{\psi}_A \gamma^\mu + i\bar{\psi}_B \gamma^\mu \star i A_{\mu AB}) - im\bar{\psi}_A = 0.$$

Again, Eq. (2.47) leads to a consistency relation that can be verified with the help of the field equations. It is, however, important that the representation space for the field ψ and the vector potential $A_{\mu AB}$ are the same. The representation space of the matter fields determines the space for the gauge potentials.

We conclude that there is a conserved current:

$$j^\rho_{AB} = i[A_\mu \stackrel{\star}{,} F^{\mu\rho}]_{AB} - \gamma^\rho_{\alpha\beta} \psi_{\beta A} \star \bar{\psi}_{\alpha B}. \tag{2.49}$$

We were again able to find a conserved current as a consequence of a deformed symmetry. Even if we put the vector potential to zero there remains the part from the matter field. There are conservation laws due to a deformed symmetry. It is remarkable that we have found conserved currents in the twisted theory as well. In the *undeformed* theory we can derive them with the help of the Noether theorem. In the deformed theory this is not possible. Nevertheless the property that a theory has a conserved current is preserved by a deformation. This is an important step to convince ourselves that a deformed gauge theory has properties close to what we need for physics.

2.6 Examples

1) Maxwell equations

We start from the simplest gauge theory based on U(1) and describing gauge fields only. We proceed schematically. The transformation law of the gauge field A_μ:

$$\delta^\star_\alpha A_\mu = \partial_\mu \alpha. \tag{2.50}$$

The covariant derivative:

$$D^\star_\mu = \partial_\mu - iA_\mu \star. \tag{2.51}$$

The field strength tensor:

$$F_{\mu\nu} = [D^\star_\mu \stackrel{\star}{,} D^\star_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu]. \tag{2.52}$$

The Lagrangian:

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu}. \tag{2.53}$$

The field equations:

$$\partial^\mu F_{\mu\nu} - i[A^\mu \overset{\star}{,} F_{\mu\nu}] = 0. \tag{2.54}$$

Consistency equations:

$$\partial^\nu [A^\mu \overset{\star}{,} F_{\mu\nu}] = 0. \tag{2.55}$$

A schematic proof of the consistency condition:

$$[\partial^\nu A^\mu \overset{\star}{,} F_{\mu\nu}] + [A^\mu \overset{\star}{,} \partial^\nu F_{\mu\nu}] = \tag{2.56}$$

$$= \frac{i}{2}[[A^\nu \overset{\star}{,} A^\mu] \overset{\star}{,} F_{\mu\nu}] + i[A^\mu \overset{\star}{,} [A^\nu \overset{\star}{,} F_{\mu\nu}]]. \tag{2.57}$$

We have used the field equations and the fact that $[F_{\mu\nu} \overset{\star}{,} F^{\mu\nu}] = 0$. The terms left can now be rearranged

$$[[A^\nu \overset{\star}{,} A^\mu] \overset{\star}{,} F_{\mu\nu}] + [[A^\mu \overset{\star}{,} F_{\mu\nu}] \overset{\star}{,} A^\nu] + [[F_{\mu\nu} \overset{\star}{,} A^\nu] \overset{\star}{,} A^\mu] \tag{2.58}$$

and vanish due to the Jacobi identity.

We found a conserved current:

$$j_\nu = [A^\mu \overset{\star}{,} F_{\mu\nu}], \qquad \partial_\nu j^\nu = 0. \tag{2.59}$$

2) Electrodynamics with one charged spinor field

Transformation law of the gauge field and the spinor field:

$$\delta^\star_\alpha \psi = i\alpha\psi, \quad \delta^\star_\alpha A_\mu = \partial_\mu \alpha. \tag{2.60}$$

Covariant derivative:

$$D^\star_\mu = (\partial_\mu - iA_\mu \star), \quad D^\star_\mu \psi = (\partial_\mu - iA_\mu \star)\psi. \tag{2.61}$$

Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \overset{\star}{,} A_\nu]. \tag{2.62}$$

Lagrangian:

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} + \bar\psi \star \gamma^\mu (i\partial_\mu \psi + A_\mu \star \psi) - m\bar\psi \star \psi. \tag{2.63}$$

Field equations:

$$\begin{aligned} &\partial_\mu F^{\mu\rho} + i[A_\mu \overset{\star}{,} F^{\rho\mu}] + \gamma^\rho \psi \star \bar\psi = 0, \\ &\gamma^\mu (\partial_\mu \psi) - i\gamma^\mu A_\mu \star \psi + im\psi = 0, \\ &(\partial_\mu \bar\psi)\gamma^\mu + i\bar\psi \gamma^\mu \star A_\nu - im\bar\psi = 0. \end{aligned} \tag{2.64}$$

Consistency condition:

$$\partial_\rho\left([A_\mu \stackrel{\star}{,} F^{\rho\mu}] + \gamma^\rho \psi \star \bar{\psi}\right) = 0. \tag{2.65}$$

Proof: As before, the spinor terms have to be added in the current and the field equations.

Current:

$$j^\rho = [A_\nu \stackrel{\star}{,} F^{\rho\nu}] + \gamma^\rho \psi \star \bar{\psi}, \quad \partial_\nu j^\nu = 0. \tag{2.66}$$

3) Electrodynamics with several charged fields

We try to formulate a model with one vector potential and differently charged matter fields as we do in the undeformed situation. This amounts to introduce an $U(1)$ gauge-invariant action for the gauge potential and for the matter fields.

Let us consider the part of the vector potential first.

The transformation law is

$$\delta_\alpha^\star A_\mu = \partial_\mu \alpha. \tag{2.67}$$

The covariant derivative

$$D_\mu^\star = (\partial_\mu - iA_\mu \star) \tag{2.68}$$

gives the following field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu]. \tag{2.69}$$

As an invariant Lagrangian we choose

$$\mathscr{L}_A = -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu}. \tag{2.70}$$

Next we consider the matter fields ψ^r with charges g_r, $r = 1, \ldots, n$. They transform as follows:

$$\delta_\alpha^\star \psi^r = ig_r \alpha \psi^r. \tag{2.71}$$

The covariant derivative depends on the charge of the field it acts on:

$$D_\mu^\star \psi^r = (\partial_\mu - ig_r A_\mu \star)\psi^r. \tag{2.72}$$

The $U(1)$ gauge-invariant action can be chosen as follows:

$$\mathscr{L}_\psi = \sum_r \bar{\psi}^r \star \gamma^\mu \left(i(\partial_\mu \psi) + g_r A_\mu \star \psi^r\right) - m_r \bar{\psi}^r \star \psi^r. \tag{2.73}$$

As the total Lagrangian we take the sum

$$\mathscr{L} = \mathscr{L}_A + \mathscr{L}_\psi. \tag{2.74}$$

It is $U(1)$ gauge invariant and it is a deformation of the usual electrodynamics with different charged fields. This Lagrangian now leads to the field equations:

$$\begin{aligned}
&\partial_\mu F^{\mu\rho} + i[A_\mu \stackrel{\star}{,} F^{\rho\mu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r = 0,\\
&\gamma^\mu (\partial_\mu \psi) - i g_r \gamma^\mu A_\mu \star \psi + i m_r \psi^r = 0,\\
&\partial_\mu \bar{\psi}^r \gamma^\mu + i \bar{\psi}^r \gamma^\mu \star g^r A_\nu - i m_r \bar{\psi}^r = 0.
\end{aligned} \tag{2.75}$$

The first of these equations gives rise to a consistency condition:

$$\partial_\rho \left(i[A_\nu \stackrel{\star}{,} F^{\rho\nu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r \right) = 0. \tag{2.76}$$

From a direct calculation, using the field equations, follows:

$$\partial_\rho \left(i[A_\nu \stackrel{\star}{,} F^{\rho\nu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r \right) \tag{2.77}$$

$$= -\sum_r (g_r^2 - g_r)[A_\mu \stackrel{\star}{,} \gamma^\mu \psi^r \star \bar{\psi}^r]. \tag{2.78}$$

The consistency condition is only satisfied if $g_r = g_r^2$ or $g_r = 1$. With one vector potential we can in a U(1) model only describe particles with one charge. There can be an arbitrary number of matter fields with this charge. This is different from the usual undeformed situation. There the commutator in (2.69) vanishes and does not give rise to an inconsistency.

This is not surprising, we forgot that the vector potential has at least to be enveloping algebra valued. This is demonstrated in the next example.

4) Electrodynamics of a positive and a negative charged matter field

The gauge group is supposed to be $U(1)$ and the matter fields are in the multiplet that transforms as follows:

$$\delta_\alpha^\star \psi = i\alpha Q \psi, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.79}$$

As outlined in Sect. 2.5, the gauge potential has to be in the same representation of the enveloping algebra as the matter fields are.

The enveloping algebra has two elements:

$$I \text{ and } Q, \quad Q^2 = 1. \tag{2.80}$$

We generalize the transformation law (2.79) to be enveloping algebra valued

$$\delta_\Lambda \psi = i\Lambda \psi, \qquad \Lambda = \lambda_0(x) I + \lambda_1(x) Q. \tag{2.81}$$

The vector potential A_μ has the analogous decomposition

$$\mathscr{A}_\mu = A_\mu(x) I + B_\mu(x) Q. \tag{2.82}$$

The covariant derivative is

$$D^\star_\mu \psi = (\partial_\mu - i\mathscr{A}_\mu \star)\psi = (\partial_\mu - iA_\mu(x)\star I - iB_\mu(x)\star Q)\psi. \tag{2.83}$$

The field strength can also be decomposed in the enveloping algebra

$$\mathscr{F}_{\mu\nu} = F_{\mu\nu} I + G_{\mu\nu} Q. \tag{2.84}$$

From the definition of the field strength

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - i[\mathscr{A}_\mu \stackrel{\star}{,} \mathscr{A}_\nu], \tag{2.85}$$

follows

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu] - i[B_\mu \stackrel{\star}{,} B_\nu], \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu - i[A_\mu \stackrel{\star}{,} B_\nu] - i[B_\mu \stackrel{\star}{,} A_\nu]. \end{aligned} \tag{2.86}$$

The matter fields couple to the vector potential via the covariant derivative

$$\begin{aligned} D^\star_\mu \psi &= (\partial_\mu - i\mathscr{A}_\mu \star)\psi \\ &= (\partial_\mu - iA_\mu(x)\star I - iB_\mu(x)\star Q)\psi. \end{aligned} \tag{2.87}$$

This leads to the Lagrangian

$$\mathscr{L} = -\frac{1}{4}\mathscr{F}^{\mu\nu}\star\mathscr{F}_{\mu\nu} + \bar{\psi}\star\gamma^\mu\left(i(\partial_\mu\psi) + \mathscr{A}_\mu\star\psi\right) - m\bar{\psi}\star\psi \tag{2.88}$$

and the field equations

$$\begin{aligned} \frac{\delta\mathscr{L}}{\delta A_\rho}: &\quad \partial_\mu F^{\mu\rho} + i[A_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[B_\mu \stackrel{\star}{,} G^{\rho\mu}] + i\gamma^\rho\psi\star\bar{\psi} = 0, \\ \frac{\delta\mathscr{L}}{\delta B_\rho}: &\quad \partial_\mu G^{\mu\rho} + i[B_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[A_\mu \stackrel{\star}{,} G^{\rho\mu}] + i\gamma^\rho\psi_A\star\bar{\psi}_B Q^{AB} = 0, \\ \frac{\delta\mathscr{L}}{\delta\bar{\psi}}: &\quad \gamma^\mu(\partial_\mu\psi) - i\gamma^\mu\mathscr{A}_\mu\star\psi + m\psi = 0, \\ \frac{\delta\mathscr{L}}{\delta\psi}: &\quad \partial_\mu\bar{\psi}\gamma^\mu + i\bar{\psi}\gamma^\mu\star\mathscr{A}_\mu - m\bar{\psi} = 0. \end{aligned} \tag{2.89}$$

We obtain two consistency equations that render two transformation laws, in agreement with the extended symmetry (2.81)

$$j^\rho_A = i[A_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[B_\mu \stackrel{\star}{,} G^{\rho\mu}] + \gamma^\rho\psi_A\star\bar{\psi}_A, \tag{2.90}$$

with

$$\partial_\rho j^\rho_A = 0 \tag{2.91}$$

and

$$j^\rho_B = i[B_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[A_\mu \stackrel{\star}{,} G^{\rho\mu}] - i\gamma^\rho\psi_A\star\bar{\psi}_B Q^{AB}. \tag{2.92}$$

We learn that the deformed gauge theory leads to a theory with a larger symmetry structure, the enveloping algebra structure. This structure survives in the limit $\theta \to 0$. We find the corresponding conservation laws and gauge transformations needed for a consistent gauge theory.

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Chapter 3
Einstein Gravity on Deformed Spaces

Julius Wess

A differential calculus, differential geometry, and the Einstein gravity theory are studied on noncommutative spaces. Noncommutativity is formulated in the star product formalism. The basis for the gravity theory is the infinitesimal algebra of diffeomorphisms. Considering the corresponding Hopf algebra we find that the deformed gravity is based on a deformation of the Hopf algebra.

3.1 Introduction

Gravity theories and differential geometry have been developed on differential manifolds where the functions form an algebra by pointwise multiplication:

$$\mu\{f \otimes g\} = f \cdot g. \tag{3.1}$$

In this chapter I want to show that these theories can be generalized by deforming this product [1, 2]. There are many deformations of the pointwise product to a star product [3–6]; the simplest and most discussed is the Moyal product [7, 8] which is introduced in Chap. 1:

$$\mu_\star\{f \otimes g\} \equiv f \star g = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\}. \tag{3.2}$$

This product can be shown to be associative but it is not commutative. It is defined for C^∞ functions in general as a formal power series in $\theta^{\rho\sigma}$.[1] Evaluated on the functions x^μ and x^ν (3.1) yields

$$x^\mu \star x^\nu - x^\nu \star x^\mu \equiv [x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}. \tag{3.3}$$

[1] Remember that like in the previous chapter the deformation parameter h is absorbed in $\theta^{\rho\sigma}$.

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These, mathematically, are the canonical commutation relations of quantum mechanics but here we postulate them for the configuration space.

A differential calculus on noncommutative spaces has been developed [9–11]. Considering differentiation as a map from the space of functions to the space of functions

$$\partial_\rho : \qquad f \mapsto \partial_\rho f, \tag{3.4}$$

it can be generalized to an algebra map.[2]

Recognizing that $f \star g$ is a function, again one finds the deformed Leibniz rule:

$$\partial_\rho (f \star g) = (\partial_\rho f) \star g + f \star (\partial_\rho g) + f(\partial_\rho \star) g. \tag{3.5}$$

In the case of the Moyal product the $\star$-operation is x-independent and we obtain the usual Leibniz rule.[3] To indicate that the derivative now is a map from the deformed algebra of functions $\mathscr{A}_x^\star$ to the deformed algebra of functions $\mathscr{A}_x^\star$ we denote it by $\partial^\star$

$$\begin{aligned} \partial_\rho^\star f &\equiv \partial_\rho f \\ \partial_\rho^\star (f \star g) &= (\partial_\rho^\star f) \star g + f \star (\partial_\rho^\star g). \end{aligned} \tag{3.6}$$

These equations establish a well-defined differential calculus on the deformed space of functions. They allow us to consider $\partial_\rho^\star$ as a linear operator with the properties:

$$\partial_\rho^\star \partial_\sigma^\star = \partial_\sigma^\star \partial_\rho^\star \tag{3.7}$$

and

$$\begin{aligned} \partial_\rho^\star f &= (\partial_\rho^\star f) + f \partial_\rho^\star \\ &= (\partial_\rho f) + f \partial_\rho. \end{aligned} \tag{3.8}$$

The following treatment of deformed differential geometry will be based on Eqs. (3.2), (3.7), and (3.8). It is only Eq. (3.4) that defines the ordinary derivative of a function that has to be used as an a priori input. The generalization to the deformed situation is essentially algebraic in nature.

3.2 Differential operators

We now consider the extension of the algebra of functions (deformed or undeformed) by the algebra of derivatives. From the Leibniz rule (3.8) follows that there is a basis where the derivatives are all at the right-hand side of the functions. An element of the extended algebra in this basis we call a differential operator [1].

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[2] This was discussed in Chap. 1 and will be mentioned again in Chap. 4, Sect. 4.2.

[3] In this chapter we only consider the θ-deformed space which was introduced and discussed in some detail in Chap. 1.

On the undeformed algebra of functions we write

$$\mathscr{D}_{\{d\}} = \sum_{r\geq 0} d_r^{\rho_1\ldots\rho_r} \partial_{\rho_1}\ldots\partial_{\rho_r}. \tag{3.9}$$

On the deformed algebra of functions we write

$$\mathscr{D}^\star_{\{d\}} = \sum_{r\geq 0} d_r^{\rho_1\ldots\rho_r} \partial^\star_{\rho_1}\ldots\partial^\star_{\rho_r}. \tag{3.10}$$

A differential operator is characterized by the coefficient functions d_r. This is indicated by $\{d\}$. We shall frequently omit this indication and write $\mathscr{D}$ for a differential operator, with coefficient function d_r and $\mathscr{D}'$ for d'_r.

Differential operators can be multiplied using the algebraic properties (3.1) or (3.2) in the deformed case and the relations (3.7) and (3.8).

The product can always be expressed in terms of differential operators by reordering it with the help of the Leibniz rule. In this sense the differential operators form an algebra in both cases, deformed and undeformed. As in Chap. 1, we call $\mathscr{A}\mathscr{D}_{\{d\}}$ the undeformed algebra of differential operators and $\mathscr{A}^\star\mathscr{D}^\star_{\{d\}}$ the deformed one[4]. There is a map from the operators $\mathscr{A}\mathscr{D}_{\{d\}}$ to the operators $\mathscr{A}^\star\mathscr{D}^\star_{\{d\}}$ that is an algebra isomorphism

$$X^\star : \mathscr{A}\mathscr{D}_{\{d\}} \to \mathscr{A}^\star\mathscr{D}^\star_{\{d\}} . \tag{3.11}$$

To define this map we let the differential operators act on a function g:

$$\mathscr{D}g = \sum_{r\geq 0} d_r^{\rho_1\ldots\rho_r} \left(\partial_{\rho_1}\ldots\partial_{\rho_r} g\right), \tag{3.12}$$

or

$$\mathscr{D}^\star \star g = \sum_{r\geq 0} d_r^{\rho_1\ldots\rho_r} \star \left(\partial^\star_{\rho_1}\ldots\partial^\star_{\rho_r} g\right). \tag{3.13}$$

Given an operator $\mathscr{D}$ we construct a new operator $X^\star_{\mathscr{D}}$ such that its star action equals the undeformed action of the initial operator $\mathscr{D}$. For any function g,

$$X^\star_{\mathscr{D}} \star g = \mathscr{D}g. \tag{3.14}$$

Because $X^\star_{\mathscr{D}} \star g$ is a function we can apply $X^\star_{\widetilde{\mathscr{D}}}$ once more:

$$X^\star_{\widetilde{\mathscr{D}}} \star (X^\star_{\mathscr{D}} \star g) = (X^\star_{\widetilde{\mathscr{D}}} \star X^\star_{\mathscr{D}}) \star g. \tag{3.15}$$

The left-hand side can also be evaluated by using (3.14) consecutively:

$$\begin{aligned} X^\star_{\widetilde{\mathscr{D}}} \star (X^\star_{\mathscr{D}} \star g) &= X^\star_{\widetilde{\mathscr{D}}} \star (\mathscr{D}g) = \widetilde{\mathscr{D}}\mathscr{D}g \\ &= X^\star_{(\widetilde{\mathscr{D}}\mathscr{D})} \star g. \end{aligned} \tag{3.16}$$

[4] For example, the deformed product of the differential operators given by $\mathscr{D} = d^\rho\partial_\rho$ and $\mathscr{D}' = d'^{\sigma_1\sigma_2}\partial_{\sigma_1}\partial_{\sigma_2}$ is $\mathscr{D}\star\mathscr{D}' = d^\rho \star (\partial_\rho d'^{\sigma_1\sigma_2})\partial_{\sigma_1}\partial_{\sigma_2} + d^\rho \star d'^{\sigma_1\sigma_2}\partial_\rho\partial_{\sigma_1}\partial_{\sigma_2}$.

We conclude

$$X^\star_{\widetilde{\mathscr{D}}} \star X^\star_{\mathscr{D}} = X^\star_{(\widetilde{\mathscr{D}}\mathscr{D})}. \tag{3.17}$$

Multiplying g pointwise with a function f forms a subalgebra of $\mathscr{D}$. We shall now construct the operator $X^\star_f$ explicitly for this case starting from (3.1):

$$\begin{aligned} f \cdot g = \mu\{f \otimes g\} &= \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} f \otimes g\} \\ &= \mu_\star\{e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} f \otimes g\}. \end{aligned} \tag{3.18}$$

More explicitly:

$$X^\star_f = \sum_{r\geq 0} \frac{1}{r!}\left(-\frac{i}{2}\right)^r \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_r\sigma_r}(\partial_{\rho_1}\ldots\partial_{\rho_r}f)\partial^\star_{\sigma_1}\ldots\partial^\star_{\sigma_r}. \tag{3.19}$$

This operator has the properties

$$X^\star_f \star g = f \cdot g \tag{3.20}$$

and

$$X^\star_f \star X^\star_g = X^\star_{fg}. \tag{3.21}$$

It is given by a power series in θ that at zeroth order is the identity. We thus have $X^\star_f = f + \mathscr{O}(\theta)$ and $X^\star$ is therefore an invertible map.

The algebra of functions with pointwise multiplication (i.e., the subalgebra of undeformed zeroth-order differential operators) is mapped into the star-deformed algebra of the differential operators $X^\star_f$. This is a subalgebra of the deformed algebra of differential operators $\mathscr{A}^\star\mathscr{D}^\star_{\{d\}}$.

The Lie algebra of infinitesimal (local) diffeomorphisms on $\mathscr{A}_x$ is generated by vector fields

$$\begin{aligned} \xi &= \xi^\mu(x)\partial_\mu, \\ [\xi,\eta] &= \left(\xi^\mu(\partial_\mu\eta^\rho) - \eta^\mu(\partial_\mu\xi^\rho)\right)\partial_\rho \\ &= (\xi\times\eta)^\rho\partial_\rho \equiv \xi\times\eta. \end{aligned} \tag{3.22}$$

The commutator of two vector fields is a vector field again. This is not the case for the star commutator because the $\star$-product of two functions does not commute. The differential operators $X^\star_\xi$, however, will form an algebra under the star commutator:

$$[X^\star_\xi \stackrel{\star}{,} X^\star_\eta] = X^\star_{\xi\times\eta}. \tag{3.23}$$

This follows from (3.17).

The operator $X^\star_\xi$ satisfies

$$X^\star_\xi \star g = \xi g, \tag{3.24}$$

for any function g, and is easily constructed starting from (3.22)

$$\xi g = \xi^\mu\partial_\mu g = \xi^\mu(\partial_\mu g) = X^\star_{\xi^\mu} \star (\partial^\star_\mu g). \tag{3.25}$$

Thus, we find

$$X^\star_\xi = X^\star_{\xi^\mu} \star \partial^\star_\mu. \tag{3.26}$$

Again the usual Lie algebra of infinitesimal diffeomorphisms (first-order differential operators with Lie bracket given by the commutator in $\mathscr{A}\mathscr{D}_{\{d\}}$) is mapped into the Lie algebra of the differential operators $X^\star_\xi$, where the bracket is given by the commutator in $\mathscr{A}^\star\mathscr{D}^\star_{\{d\}}$,

$$\begin{aligned} \xi &\mapsto X^\star_\xi \\ [X^\star_\xi \stackrel{\star}{,} X^\star_\eta] &= X^\star_{\xi\times\eta}. \end{aligned} \tag{3.27}$$

This is the starting point for the construction of a tensor calculus on tensor fields.

3.3 Tensor fields

The classical theory of gravity is based on invariance under coordinate transformations.[5] This leads to the concept of scalar, vector, and tensor fields that transform under infinitesimal general coordinate transformation as follows:

$$\begin{array}{ll} \text{scalar:} & \delta_\xi \phi(x) = -\xi\phi, \\ \text{covariant vector:} & \delta_\xi V_\mu(x) = -\xi V_\mu - (\partial_\mu \xi^\rho)V_\rho, \\ \text{contravariant vector:} & \delta_\xi V^\mu(x) = -\xi V^\mu + (\partial_\rho \xi^\mu)V^\rho \end{array} \tag{3.29}$$

and similarly for other tensor fields.[6] Note that in (3.29) the variation δ_ξ stands for $\delta_\xi \phi = \phi'(x) - \phi(x)$.

The concept of coordinate transformations is difficult to generalize to deformed spaces, but the transformation laws of fields are representations of the Lie algebra of infinitesimal diffeomorphisms, that we just learned how to deform. Recalling the transformation law (3.24) of functions under deformed infinitesimal diffeomorphisms we define the following transformation laws of fields under the deformed algebra of diffeomorphisms:

$$\begin{aligned} \delta^\star_\xi \phi &= -X^\star_\xi \star \phi = -\xi\phi, \\ \delta^\star_\xi V_\mu &= -X^\star_\xi \star V_\mu - X^\star_{(\partial_\mu \xi^\rho)} \star V_\rho = -\xi V_\mu - (\partial_\mu \xi^\rho)V_\rho, \\ \delta^\star_\xi V^\mu &= -X^\star_\xi \star V^\mu + X^\star_{(\partial_\rho \xi^\mu)} \star V^\rho = -\xi V^\mu + (\partial_\rho \xi^\mu)V^\rho \end{aligned} \tag{3.31}$$

and similarly for other tensor fields.

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[5] Infinitesimal coordinate transformations are given by

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{3.28}$$

with infinitesimal $\xi^\mu(x)$.

[6] For example, for a covariant tensor of rank 2 we have

$$\delta_\xi T_{\mu\nu}(x) = -\xi T_{\mu\nu} - (\partial_\mu \xi^\rho)T_{\rho\nu} - (\partial_\nu \xi^\rho)T_{\mu\rho}. \tag{3.30}$$

To construct Lagrangians we have to know how the $\star$-product of fields transforms. These products should transform as tensor fields again, e.g., the $\star$-product of two scalar fields should transform as a scalar field again

$$\begin{aligned} \delta_\xi^\star(\phi\star\psi) &= -X_\xi^\star\star(\phi\star\psi) \\ &= -\xi(\phi\star\psi). \end{aligned} \tag{3.32}$$

A direct calculation shows that this is identical to

$$\delta_\xi^\star(\phi\star\psi) = -\mu_\star\{\mathscr{F}\Delta(\xi)\mathscr{F}^{-1}\phi\otimes\psi\}. \tag{3.33}$$

Here $\Delta(\xi)$ is the usual comultiplication Δ on the vector field ξ,

$$\Delta(\xi) = \xi\otimes 1 + 1\otimes\xi\ . \tag{3.34}$$

$\mathscr{F}$ is called a twist and it is the element

$$\mathscr{F} = e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}. \tag{3.35}$$

The right-hand side of (3.32) and (3.33) can be calculated in a power series expansion in θ and will be found to be the same.

The advantage of the expression (3.33) is that it links to the formalism of constructing new Hopf algebras (symmetries) by deforming via a twist existing ones [12–18], see Chap. 8 for a short introduction and Chap. 7 for more details concerning Hopf algebras. Many results are known there [19, 20]. We first have to establish that the twist $\mathscr{F}$ defined in (3.35) really satisfies the conditions for a twist. This is the case (see a proof in (8.8) and (8.9)).

Then we can use the twist to deform the Leibniz rule for arbitrary tensor fields. The procedure is as follows:

First consider the coproduct (also called comultiplication) for the undeformed transformations

$$\Delta(\delta_\xi) = \delta_\xi\otimes 1 + 1\otimes\delta_\xi, \tag{3.36}$$

where the variations δ_ξ are expressed by differential operators such that

$$\delta_\xi(\phi\otimes\psi) = (\delta_\xi\phi)\otimes\psi + \phi\otimes(\delta_\xi\psi) \tag{3.37}$$

for any two tensor fields, this follows from the usual Leibniz rule of infinitesimal variations. This coproduct can be twisted

$$\Delta_{\mathscr{F}}(\delta_\xi) = \mathscr{F}\Delta(\delta_\xi)\mathscr{F}^{-1}. \tag{3.38}$$

Finally define the deformed Leibniz rule

$$\delta_\xi^\star(\phi\star\psi) = \mu_\star\{\Delta_{\mathscr{F}}(\delta_\xi)\phi\otimes\psi\}. \tag{3.39}$$

This is not limited to scalar fields but applies also to the $\star$-product of generic tensor fields $\phi = T^{\mu_1\dots\mu_m}_{\nu_1\dots\nu_n}$ and $\psi = T^{\sigma_1\dots\sigma_s}_{\rho_1\dots\rho_r}$.

We can convince ourselves that this Leibniz rule has the properties demanded at the beginning of this section, i.e., that $\star$-products of tensor fields transform as tensor fields (cf. (3.33) for scalar fields).

The deformed coproduct $\Delta_{\mathscr{F}}$ defines a new Hopf algebra. As a Hopf algebra the algebra of infinitesimal diffeomorphisms is deformed!

Note

More explicitly the deformed Leibniz rule reads

$$\begin{aligned}
\delta^\star_\xi(\phi\star\psi) &= \mu_\star\{\Delta_{\mathscr{F}}(\delta_\xi)\phi\otimes\psi\}\\
&= (\delta^\star_\xi\phi)\star\psi+\phi\star(\delta^\star_\xi\psi)\\
&\quad+\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\right)^n\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\{\delta^\star_{(\partial_{\rho_1}\dots\partial_{\rho_n}\xi)}\phi\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)\\
&\quad+(\partial_{\rho_1}\dots\partial_{\rho_n}\phi)\star\delta^\star_{(\partial_{\sigma_1}\dots\partial_{\sigma_n}\xi)}\psi\}.
\end{aligned}\tag{3.40}$$

The calculation in order to establish the second equality is very similar to the one we performed in order to find the Leibniz rule for the deformed gauge transformations in Chap. 1, Sect. 1.6.

Let us see in an example that by using the deformed comultiplication the $\star$-product of two tensor fields $T^{\mu_1\dots\mu_m}_{\nu_1\dots\nu_n}\star T^{\sigma_1\dots\sigma_s}_{\rho_1\dots\rho_r}$ transforms like the tensor field $T^{\mu_1\dots\mu_m\sigma_1\dots\sigma_s}_{\nu_1\dots\nu_n\rho_1\dots\rho_r}$. Consider the $\star$-product of a scalar and a vector field

$$\begin{aligned}
\delta^\star_\xi(\phi\star V_\mu) &= \mu_\star\left\{e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\left(\delta^\star_\xi\otimes 1+1\otimes\delta^\star_\xi\right)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}(\phi\otimes V_\mu)\right\}\\
&= \mu_\star\Big\{\delta^\star_\xi\phi\otimes V_\mu+\phi\otimes\delta^\star_\xi V_\mu\\
&\quad-\frac{i}{2}\theta^{\rho\sigma}\left(\left([\partial_\rho,\delta^\star_\xi]\phi\right)\otimes(\partial_\sigma V_\mu)+(\partial_\rho\phi)\otimes\left([\partial_\sigma,\delta^\star_\xi]V_\mu\right)\right)+\mathscr{O}(\theta^2)\Big\}\\
&= \delta^\star_\xi\phi\star V_\mu+\phi\star\delta^\star_\xi V_\mu\\
&\quad-\frac{i}{2}\theta^{\rho\sigma}\left((\delta^\star_{\partial_\rho\xi}\phi)\star(\partial_\sigma V_\mu)+(\partial_\rho\phi)\star(\delta^\star_{\partial_\sigma\xi}V_\mu)\right)+\mathscr{O}(\theta^2)\\
&= -\xi(\phi\star V_\mu)-(\partial_\mu\xi^\lambda)(\phi\star V^\lambda)\\
&= -X^\star_\xi\star(\phi\star V_\mu)-X^\star_{(\partial_\mu\xi^\lambda)}\star(\phi\star V_\lambda).
\end{aligned}\tag{3.41}$$

In the first line the definition of the deformed Leibniz rule is used and expanded to the first order in the deformation parameter $\theta^{\rho\sigma}$. Then all the $\star$-products were expanded and terms were collected in such a way that the line above the last is obtained. In the last line the result is rewritten in terms of higher order differential operators $X^\star$. Comparing (3.41) with the transformation law of a vector field (3.31) we see that $\phi\star V_\mu$ indeed transforms as a vector field.

3.4 Einstein–Hilbert gravity

The Einstein–Hilbert theory of gravity can now be constructed following its presentation in a textbook.

1) Covariant derivatives

The covariant derivative of a tensor field should again transform as a tensor field. This can be done with the help of a connection Γ. For a covariant vector field

$$D_\mu \star V_\nu = \partial^\star_\mu V_\nu - \Gamma^\alpha_{\mu\nu} \star V_\alpha. \tag{3.42}$$

For (3.42) to be a covariant derivative

$$\begin{aligned} \delta^\star_\xi D_\mu \star V_\nu &= -X^\star_\xi \star (D_\mu \star V_\nu) - X^\star_{(\partial_\mu \xi^\rho)}(D_\rho \star V_\nu) - X^\star_{(\partial_\nu \xi^\rho)}(D_\mu \star V_\rho) \\ &= -\xi(D_\mu \star V_\nu) - (\partial_\mu \xi^\rho)(D_\rho \star V_\nu) - (\partial_\nu \xi^\rho)(D_\mu \star V_\rho) \end{aligned} \tag{3.43}$$

the connection has to transform as follows[7]:

$$\delta^\star_\xi \Gamma^\alpha_{\mu\nu} = -X^\star_\xi \star \Gamma^\alpha_{\mu\nu} - X^\star_{(\partial_\mu \xi^\rho)} \star \Gamma^\alpha_{\rho\nu} - X^\star_{(\partial_\nu \xi^\rho)} \star \Gamma^\alpha_{\mu\rho} + X^\star_{(\partial_\rho \xi^\alpha)} \star \Gamma^\rho_{\mu\nu} - \partial_\mu \partial_\nu \xi^\alpha. \tag{3.44}$$

This can easily be generalized to arbitrary tensor fields.[8]

2) Curvature and torsion

The curvature and torsion tensors can be defined as usual [21]

$$[D_\mu \stackrel{\star}{,} D_\nu] \star V_\rho = R_{\mu\nu\rho}{}^\sigma \star V_\sigma + T_{\mu\nu}{}^\alpha \star D_\alpha \star V_\rho. \tag{3.46}$$

They can be expressed in terms of the connection:

$$R_{\mu\nu\rho}{}^\sigma = \partial^\star_\nu \Gamma^\sigma_{\mu\rho} - \partial^\star_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\beta_{\nu\rho} \star \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} \star \Gamma^\sigma_{\nu\beta}, \tag{3.47}$$

$$T_{\mu\nu}{}^\alpha = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}. \tag{3.48}$$

From the transformation law of the connection (3.44) follows that curvature and torsion transform like tensors if the deformed Leibniz rule (3.39) is used. From now

[7] Hint, use (3.43), the deformed Leibniz rule (3.40), and the fact that $\delta^\star_\xi(\partial_\mu V_\nu) = \partial_\mu(\delta^\star_\xi V_\nu)$.

[8] In the commutative case this generalization is done via the Leibniz rule for the covariant derivative. In the deformed case the usual Leibniz rule for the covariant derivative does not hold, check for example $D_\lambda \star (V^\mu \star V_\mu)$. Instead we define the covariant derivative on a tensor field to be

$$\begin{aligned} D_\lambda \star T^{\nu_1 \ldots \nu_r}_{\mu_1 \ldots \mu_p} &= \partial_\lambda T^{\nu_1 \ldots \nu_r}_{\mu_1 \ldots \mu_p} - \Gamma^\alpha_{\lambda\mu_1} \star T^{\nu_1 \ldots \nu_r}_{\alpha \ldots \mu_p} - \cdots - \Gamma^\alpha_{\lambda\mu_p} \star T^{\nu_1 \ldots \nu_r}_{\mu_1 \ldots \alpha} \\ &\quad + \Gamma^{\nu_1}_{\lambda\alpha} \star T^{\alpha \ldots \nu_r}_{\mu_1 \ldots \mu_p} + \cdots + \Gamma^{\nu_r}_{\lambda\alpha} \star T^{\nu_1 \ldots \alpha}_{\mu_1 \ldots \mu_p}. \end{aligned} \tag{3.45}$$

on we assume for simplicity that $T_{\mu\nu}{}^{\alpha} = 0$, i.e., that the connection $\Gamma^{\sigma}_{\mu\rho}$ is symmetric in its lower indices.

3) Metric tensor

The relevant dynamical variable in gravity is the metric tensor. It is introduced as a covariant symmetric tensor of rank two:

$$\delta^{\star}_{\xi} G_{\mu\nu} = -X^{\star}_{\xi} \star G_{\mu\nu} - X^{\star}_{(\partial_\mu \xi^\rho)} \star G_{\rho\nu} - X^{\star}_{(\partial_\nu \xi^\rho)} \star G_{\mu\rho}. \tag{3.49}$$

For $\theta = 0$ we identify it with the usual metric field:

$$G_{\mu\nu}\Big|_{\theta=0} = g_{\mu\nu}. \tag{3.50}$$

Next we have to construct the $\star$-inverse of the metric[9]:

$$G_{\mu\nu} \star G^{\nu\rho\star} = \delta^{\rho}_{\mu}. \tag{3.51}$$

Let us first construct the $\star$-inverse of a function that is invertible in the undeformed algebra:

$$f \cdot f^{-1} = 1. \tag{3.52}$$

The star inverse $f^{-1\star}$ is defined by

$$f \star f^{-1\star} = 1. \tag{3.53}$$

It exists as a geometric series because f^{-1} exists, see Chap. 1. The additional terms are a power series in θ. To find $f^{-1\star}$ in a compact version we start from

$$f \star f^{-1} = 1 + \mathcal{O}(\theta),$$

$$\begin{aligned} \left(f \star f^{-1}\right)^{-1\star} &= \left(1 + f \star f^{-1} - 1\right)^{-1\star} \\ &= \sum_{n=0}^{\infty} \left(1 - f \star f^{-1}\right)^{n\star} \\ &= \underbrace{1}_{n=0} + \underbrace{1 - f \star f^{-1}}_{n=1} + \underbrace{1 - 2f \star f^{-1} + f \star f^{-1} \star f \star f^{-1}}_{n=2} + \cdots . \end{aligned} \tag{3.54}$$

The star at the nth power means that all the products are star products. By definition we know that

$$(f \star f^{-1}) \star (f \star f^{-1})^{-1\star} = 1. \tag{3.55}$$

The star multiplication is associative. We use this for Eq. (3.55) and write it in the form

[9] The $\star$-inverse of the metric tensor $G_{\mu\nu}$ is a function and not a differential operator.

$$f \star \left(f^{-1} \star (f \star f^{-1})^{-1\star}\right) = 1. \tag{3.56}$$

It follows that

$$f^{-1\star} = f^{-1} \star (f \star f^{-1})^{-1\star}. \tag{3.57}$$

The factor $(f \star f^{-1})^{-1\star}$ has been calculated in (3.54) as a power series expansion in f and f^{-1}. We insert this into (3.57) and find that $f^{-1\star}$ can be expressed in f and f^{-1}.

To invert the metric tensor we follow the analogous procedure. Equations

$$\begin{aligned} G_{\mu\nu} \cdot G^{\nu\rho} &= \delta_\mu^\rho, \\ G_{\mu\nu} \star G^{\nu\rho\star} &= \delta_\mu^\rho \end{aligned} \tag{3.58}$$

are the defining equations for $G^{\mu\nu}$ and $G^{\mu\nu\star}$ [10]. For $G^{\mu\nu\star}$ we find

$$G^{\mu\nu\star} = G^{\mu\rho} \star (G \star G^{-1})^{-1\star}{}_\rho{}^\nu, \tag{3.59}$$

where G and G^{-1} are short for the matrices $G_{\mu\nu}$ and $G^{\mu\nu}$, respectively. We also can show that

$$(G \star G^{-1})^{-1\star} = \sum_{n\geq 0} \left(1 - G \star G^{-1}\right)^{n\star}. \tag{3.60}$$

Because the $\star$-product is not commutative $G^{\mu\nu\star}$ will be not symmetric in μ and ν.

It can now be shown explicitly from the transformation law (3.49) for $G_{\mu\nu}$ that $G^{\mu\nu\star}$ transforms as a contravariant tensor of rank 2.

In formulating the Einstein theory we meet the determinant and the square root of the determinant. As it is more difficult to generalize the square root to a $\star$-square root we first introduce the vielbein as the "square" root of the metric tensor. It consists of four covariant vector fields $E_\mu{}^a$ that form the metric:

$$G_{\mu\nu} = \frac{1}{2}\left(E_\mu{}^a \star E_\nu{}^b + E_\nu{}^a \star E_\mu{}^b\right)\eta_{ab}. \tag{3.61}$$

As the $\star$-product is noncommutative we have symmetrized $G_{\mu\nu}$ explicitly. For the vielbein fields we demand in analogy with (3.50)

$$E_\mu{}^a\Big|_{\theta=0} = e_\mu{}^a. \tag{3.62}$$

Since $E_\mu{}^a$ transforms as a vector

$$\delta_\xi^\star E_\mu{}^a = -X_{\xi^\lambda}^\star \star (\partial_\lambda E_\mu{}^a) - X_{\partial_\mu \xi^\lambda}^\star E_\lambda{}^a, \tag{3.63}$$

[10] Both the $\star$-inverse $G^{\mu\nu\star}$ and the usual inverse $G^{\mu\nu}$ transform as contravariant tensors of rank 2. Hint: use (3.49), (3.58), and the deformed Leibniz rule in the case of $G_{\mu\nu} \star G^{\nu\rho\star}$ and the usual Leibniz rule in the case of $G_{\mu\nu}G^{\nu\rho}$. We also notice that in the zeroth order in $\theta^{\rho\sigma}$ both $G^{\mu\nu}$ and $G^{\mu\nu\star}$ are equal to the commutative metric tensor $g^{\mu\nu}$.

it follows that $G_{\mu\nu}$ transforms like a tensor, provided that the deformed Leibniz rule (3.40) is used. The $\star$-determinant of the vielbein is defined as follows:

$$E^\star = \det{}_\star E_\mu{}^a = \frac{1}{4!}\varepsilon^{\mu_1\dots\mu_4}\varepsilon_{a_1\dots a_4} E_{\mu_1}{}^{a_1} \star \cdots \star E_{\mu_4}{}^{a_4}. \tag{3.64}$$

The star on $E^\star$ and $\det_\star$ indicates that all the multiplications are $\star$-multiplications. This normalization was chosen such that

$$\det{}_\star E_\mu{}^a \Big|_{\theta=0} = \det e_\mu{}^a. \tag{3.65}$$

The second ε-tensor is necessary because the $\star$-product is noncommutative.

The important property of the determinant is that it transforms as a scalar density:

$$\delta_\xi^\star E^\star = -X_\xi^\star \star E^\star - X^\star_{(\partial_\mu \xi^\mu)} \star E^\star. \tag{3.66}$$

This is a consequence of the transformation law of the vielbein (3.63). This justifies the definition (3.64) of the $\star$-determinant.

We now have all the ingredients we need to proceed for the formulation of the Einstein–Hilbert dynamics.

4) Christoffel symbol

We demand that the covariant derivative of $G_{\mu\nu}$ vanishes:

$$D_\alpha \star G_{\beta\gamma} = \partial_\alpha^\star G_{\beta\gamma} - \Gamma^\rho_{\alpha\beta} \star G_{\rho\gamma} - \Gamma^\rho_{\alpha\gamma} \star G_{\beta\rho} = 0. \tag{3.67}$$

We permute the indices, assume that $G_{\alpha\beta}$ is symmetric, use (3.58) and obtain by following the analogous procedure of the classical case:

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha^\star G_{\beta\gamma} + \partial_\beta^\star G_{\alpha\gamma} - \partial_\gamma^\star G_{\alpha\beta}\right) \star G^{\gamma\sigma\star}. \tag{3.68}$$

The connection is entirely expressed in terms of $G_{\alpha\beta}$. In this case we call $\Gamma^\sigma_{\alpha\beta}$ the Christoffel symbol of the metric connection.

Again the transformation law of the connection (3.44) follows from the transformation law of $G_{\alpha\beta}$.

5) Ricci tensor and curvature scalar

We obtain the Ricci tensor by summing the upper index with one of the three lower indices of the curvature tensor (3.47). As the curvature tensor is antisymmetric in the first two indices we have only two choices left. Summing the second index is a deformation of the classical Ricci tensor:

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma. \tag{3.69}$$

The result $R_{\mu\nu\sigma}{}^{\sigma}$ vanishes in the commutative limit $\theta \to 0$. Nevertheless, we could add such a term to the Ricci tensor (3.69) and obtain another deformation of the classical Ricci tensor.

We see that the deformation of the classical theory is not unique. Terms that are covariant and vanish for $\theta \to 0$ are quite possible. To really make the deformation unique an additional requirement has to be added. We take simplicity and define the Ricci tensor by (3.69).[11]

The curvature scalar we define by contraction[12] with $G^{\mu\nu\star}$

$$R = G^{\mu\nu\star} \star R_{\mu\nu}. \tag{3.70}$$

Again, as $G^{\mu\nu\star}$ is not symmetric and the $\star$-product is noncommutative (3.70) is a choice.

We can now show by starting from the tensor $G_{\mu\nu}$ that the curvature scalar transforms as a scalar field

$$\delta^\star_\xi R = -X^\star_\xi \star R = -\xi^\mu(\partial_\mu R). \tag{3.71}$$

6) Lagrangian

The curvature scalar multiplied by the determinant $E^\star$ transforms like a scalar density. From (3.71) and (3.66) it follows that

$$\delta^\star_\xi(E^\star \star R) = -\partial^\star_\mu\left(X^\star_{\xi^\mu} \star (E^\star \star R)\right). \tag{3.72}$$

To define an action and the variational principle to find the field equation we have to give a definition for the integral. A possible definition is

$$\int f = \int \mathrm{d}^4x\, f, \tag{3.73}$$

i.e., we use the usual integral on commutative space (cf. Chap. 2). This integral has the trace property, which can be checked by partial integration

[11] Let us look at the symmetries of the curvature tensor (3.47) and the Ricci tensor (3.69). The curvature tensor is antisymmetric in the first two indices, but $R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho}{}^{\lambda} \star G_{\sigma\lambda}$ is not antisymmetric in ρ and σ. Also we have $R_{\mu\nu\rho\sigma} \neq R_{\rho\sigma\mu\nu}$. From this it follows that $R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma}$ is not symmetric in μ and ν. As discussed above, $R_{\mu\nu}$ is also not unique, one can add the antisymmetric part $R_{\mu\nu\sigma}{}^{\sigma}$ to it without spoiling the commutative limit.

[12] Contraction of a tensor $F_{\alpha\beta}$ is done with the noncommutative metric tensor $G^{\mu\nu\star}$. However, since the $\star$-product is not commutative expressions $G^{\mu\nu\star} \star F_{\mu\nu}$ and $F_{\mu\nu} \star G^{\mu\nu\star}$ will in general be different. An example is the definition of the scalar curvature (3.70). This definition is not unique, we could have chosen also $R = R_{\mu\nu} \star G^{\mu\nu\star}$ or a symmetrized formula $R = 1/2(G^{\mu\nu\star} \star R_{\mu\nu} + R_{\mu\nu} \star G^{\mu\nu\star})$. It is important that all these choices have the correct commutative limit (when $\theta \to 0$ they should reduce to the scalar curvature of the commutative space). Also they all have to transform as scalars under the deformed diffeomorphisms. Contraction (pairing) is also discussed in the end of Sect. 8.2.2.

$$\int d^4x \; f \star g = \int d^4x \; g \star f. \tag{3.74}$$

A suitable[13] action for a gravity theory on deformed spaces is

$$S_{\text{EH}} = \frac{1}{2} \int d^4x \; (E^{\star} \star R + \text{ c.c.}). \tag{3.75}$$

A reminder: By all the transformation laws of products of fields the deformed Leibniz rule (3.39) has to be used.

The trace property of the integral (3.73) allows us to define a variational principle, see Chaps. 2 and 4 for more details. This leads to deformed gravity equations.

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Chapter 4
Deformed Gauge Theory: Twist Versus Seiberg–Witten Approach

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In this chapter we discuss two possible ways of introducing gauge theories on noncommutative spaces. In the first approach the algebra of gauge transformations is unchanged, but the Leibniz rule is changed (compared with gauge theories on commutative space). Consistency of the equations of motion requires enveloping algebra-valued gauge fields, which leads to new degrees of freedom. In the second approach we have to go to the enveloping algebra again if we want noncommutative gauge transformations to close in the algebra. However, no new degrees of freedom appear here because of the Seiberg–Witten map. This map enables one to express noncommutative gauge parameters and fields in terms of the corresponding commutative variables.

4.1 Introduction

In previous chapters a way to deform commutative spacetime was introduced. The starting point is an abstract algebra of noncommuting coordinates. Then one uses the Poincaré–Birkhoff–Witt property to map this algebra into a space of commuting coordinates with a new noncommutative product called $\star$-product. This product can be expanded in orders of the deformation parameter which is supposed to be of the order of Planck length that is very small. In the zeroth order of the expansion the usual pointwise product is obtained. Based on this approach noncommutative gauge theories and a noncommutative theory of gravity are formulated in the previous chapters.

Gauge theories on noncommutative spaces (NC spaces) are formulated using the twist approach in Chap. 2. Especially, $U(1)$ gauge theory coupled with matter is discussed in detail. The problem of charge quantization that arises in this approach is solved by going to the enveloping algebra of $U(1)$. In this chapter we continue analyzing gauge theories on NC spaces. As in the previous chapters we work with the simplest example of noncommutative spaces, the canonically deformed space

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or θ-deformed space. In Sect. 4.2 we repeat some basic properties of this space. In Sects. 4.3 and 4.4 we present two different ways to introduce gauge theories on the θ-deformed space and we compare them. In Sect. 4.3 twisted non-abelian gauge theories [1, 2] are discussed and equations of motion for the pure Yang–Mills action are derived. Consistency of these equations enforces enveloping algebra-valued gauge fields. In Sect. 4.4 we turn to Seiberg–Witten gauge theories which are another way of introducing gauge theories on NC spaces. Seiberg–Witten gauge theories are based on the Seiberg–Witten map (SW map) between commutative and noncommutative gauge transformations and fields. This map was initially introduced for $U(N)$ gauge fields in [3], in the context of open string theory (and the zero slope limit $\alpha \to 0$ [3]). It has then been studied in the case of arbitrary gauge groups [4–8]. The SW map and the $\star$-product allow us to expand the noncommutative action order by order in the deformation parameter and to express it in terms of ordinary commutative fields. Using this approach a deformation of the standard model was constructed in [9, 10] and some new effects which do not appear in the commutative standard model were calculated in [11, 12].

Finally we end this chapter by comparing the noncommutative gauge theories obtained by using the twist approach and the Seiberg–Witten approach.

4.2 θ-deformed space

In Chap. 1 see also [13, 14] the noncommutative space $\hat{\mathscr{A}}_{\hat{x}}$ was introduced as a quotient

$$\hat{\mathscr{A}}_{\hat{x}} = \frac{\mathbb{C}[\hat{x}^0, \ldots, \hat{x}^n][[h]]}{I_{\hat{\mathscr{R}}}}. \tag{4.1}$$

Here the two-sided ideal $I_{\hat{\mathscr{R}}}$ is given by the linear span of elements

$$I_{\hat{\mathscr{R}}}: \qquad (\hat{x}\ldots\hat{x})\left([\hat{x}^\mu, \hat{x}^\nu] - i\Theta^{\mu\nu}(\hat{x})\right)(\hat{x}\ldots\hat{x}), \tag{4.2}$$

where $(\hat{x}\ldots\hat{x})$ stands for an arbitrary product of the coordinates $\hat{x}^\mu$ in the algebra $\mathbb{C}[\hat{x}^0, \ldots, \hat{x}^n][[h]]$. The algebra $\mathbb{C}[\hat{x}^0, \ldots, \hat{x}^n][[h]]$ is freely generated by $\hat{x}^\mu$ coordinates and formal power series in the parameter h are included. We also have that $\Theta^{\mu\nu}(\hat{x}) \in \mathbb{C}[\hat{x}^0, \ldots, \hat{x}^n][[h]]$ and for $h = 0$ the usual algebra of commuting coordinates is obtained.

The defining relation of the deformed space,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x}), \qquad \mu = 0, \ldots n, \tag{4.3}$$

is very general and one usually considers some special examples of it. Among them there are three very important ones:

$$\text{Canonically deformed space} \quad [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{4.4}$$

$$\text{Lie algebra deformed space} \quad [\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda, \tag{4.5}$$

$$q\text{-deformed space} \qquad \hat{x}^\mu \hat{x}^\nu = \frac{1}{q} R^{\mu\nu}{}_{\rho\sigma} \hat{x}^\rho \hat{x}^\sigma. \tag{4.6}$$

In the case of the canonically deformed space (which from now on we call θ-deformed space) [15], $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is an antisymmetric constant matrix of mass dimension -2. For Lie algebra-deformed spaces [16, 17] $C^{\mu\nu}_\lambda$ are Lie algebra structure constants of mass dimension -1. And finally, $R^{\mu\nu}{}_{\rho\sigma}$ is the dimensionless R-matrix of the quantum space [18, 19]. These three examples are important because they fulfill the Poincaré-Birkhoff-Witt (PBW) property which was mentioned in Chap. 1. This property enables us to map an arbitrary element $\hat{f}(\hat{x})$ of $\hat{\mathscr{A}}_{\hat{x}}$ to the space of commuting coordinates $\mathscr{A}_x$. First we expand $\hat{f}(\hat{x})$ in the basis of ordered monomials (we work with formal power series). Elements of this basis are labeled by $:\hat{x}^{\mu_1} \dots \hat{x}^{\mu_j}:$. In the case of symmetric ordering we have

$$\begin{aligned}
:\hat{x}^\mu: &= \hat{x}^\mu, \\
:\hat{x}^\mu \hat{x}^\nu: &= \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu), \\
&\dots, \\
:\hat{x}^{\mu_1} \dots \hat{x}^{\mu_j}: &= \frac{1}{j!} \sum_{\sigma \in S_j} \hat{x}^{\sigma(\mu_1)} \dots \hat{x}^{\sigma(\mu_j)}.
\end{aligned} \tag{4.7}$$

Then each element of the basis is mapped to the corresponding element in the space of commuting coordinates, for example, $:\hat{x}^\mu: \mapsto x^\mu$ and $:\hat{x}^\mu \hat{x}^\nu: \mapsto x^\mu x^\nu$. For the element $\hat{f}(\hat{x})$ we obtain

$$\begin{aligned}
\hat{f}(\hat{x}) \quad &= C_0 + C_{1\mu} :\hat{x}^\mu: + C_{2\mu\nu} :\hat{x}^\mu \hat{x}^\nu: + \cdots \\
\Big\downarrow \quad & \\
f(x) \quad &= C_0 + C_{1\mu} x^\mu + C_{2\mu\nu} x^\mu x^\nu + \cdots.
\end{aligned} \tag{4.8}$$

Note that $\hat{f}(\hat{x})$ in (4.8) is fully characterized by the completely symmetric coefficients $C_{\mu_1 \dots \mu_j}$.

However, multiplying two arbitrary elements $\hat{f}, \hat{g} \in \hat{\mathscr{A}}_{\hat{x}}$ gives the result which is no longer written as an expansion in basis and the elements have to be reordered. For example, we take the symmetric ordering and multiply two basis elements

$$\begin{aligned}
:\hat{x}^\mu: \cdot :\hat{x}^\nu: &= \hat{x}^\mu \hat{x}^\nu \\
&= \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu) + \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu) \\
&= :\hat{x}^\mu \hat{x}^\nu: + \frac{i}{2} \Theta^{\mu\nu}(\hat{x}).
\end{aligned} \tag{4.9}$$

In the first line we obtain a result which is not written in terms of basis elements, then we rewrite it differently. Using relations (4.3) in the last line, the result expressed in terms of basis elements follows. Once again we mention that $\Theta^{\mu\nu}(\hat{x})$ is restricted to one of the three examples (4.4)–(4.6) that fulfill PBW property.

To extend the vector space isomorphism to an algebra morphism one has to map the multiplication in the abstract algebra $\hat{\mathscr{A}}_{\hat{x}}$ to the space of commuting coordinates $\mathscr{A}_x$. Let $\hat{f}(\hat{x})$ and $\hat{g}(\hat{x})$ be two elements of $\hat{\mathscr{A}}_{\hat{x}}$. Their product is an element of $\hat{\mathscr{A}}_{\hat{x}}$:

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \hat{f} \cdot \hat{g}(\hat{x}) \in \hat{\mathscr{A}}_{\hat{x}}. \tag{4.10}$$

After reordering this element can be expanded in the chosen basis and mapped to the algebra of commuting variables $\mathscr{A}_x$

$$\hat{f} \cdot \hat{g}(\hat{x}) \mapsto f \star g(x) \in \mathscr{A}_x. \tag{4.11}$$

Its image is labeled as $f \star g(x)$ and it defines the star product ($\star$-product) of two functions. This product is bilinear and associative but noncommutative. The algebra of noncommuting coordinates $\hat{\mathscr{A}}_{\hat{x}}$ is then isomorphic to the algebra of commuting variables with the $\star$-product (instead of the usual pointwise multiplication) as multiplication, which was labeled as $\mathscr{A}_x^\star$ in the previous chapters. As we have seen in the previous chapters the $\star$-product for the θ-deformed space is given by the Moyal $\star$-product [20, 21]

$$\begin{aligned} f \star g(x) &= \mu\left(e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} f \otimes g\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\left(\partial_{\rho_1}\dots\partial_{\rho_n} f(x)\right) \\ &\qquad \left(\partial_{\sigma_1}\dots\partial_{\sigma_n} g(x)\right), \end{aligned} \tag{4.12}$$

with the pointwise multiplication μ

$$\mu(f \otimes g) = f \cdot g. \tag{4.13}$$

Once again we remind the reader that the deformation parameter h is absorbed in $\theta^{\rho\sigma}$.

Derivatives

Derivatives can be introduced as maps of the noncommutative space $\hat{\mathscr{A}}_{\hat{x}}$ to itself.[1] They are a deformation of the usual derivatives and one can make the following ansatz:

[1] The approach followed in this and in the next chapter is different than the one followed in (1.29). Nevertheless, in the case of θ-deformed space both approaches give the same result. For more complicated deformations of the commutative spacetime this will no longer be the case, see the next chapter.

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta). \tag{4.14}$$

Here $f^\mu_\rho(\hat{\partial}, \theta)$ is an operator on the algebra $\mathscr{A}_{\hat{x}}$. We suppose that it is a function only of the operators $\hat{\partial}_\rho$ and the deformation parameter $\theta^{\mu\nu}$ and not a function of coordinates $\hat{x}^\mu$. Furthermore,

$$[\hat{\partial}_\rho, \hat{\partial}_\sigma] = 0, \tag{4.15}$$

that is, derivatives commute among themselves.

Since the derivatives $\hat{\partial}_\rho$ are maps of the θ-deformed space into itself the relation (4.14) has to be consistent with (4.4)

$$\begin{aligned} \hat{\partial}_\rho \left([\hat{x}^\mu, \hat{x}^\nu] \; - \; i\, \theta^{\mu\nu}\right) = \\ \left([\hat{x}^\mu, \hat{x}^\nu] \; - \; i\theta^{\mu\nu}\right) \hat{\partial}_\rho = 0. \end{aligned} \tag{4.16}$$

That is, commuting derivative through coordinates does not lead to new commutation relations between coordinates. We calculate

$$\begin{aligned} \hat{\partial}_\rho \hat{x}^\mu \hat{x}^\nu &= ([\hat{\partial}_\rho, \hat{x}^\mu] + \hat{x}^\mu \hat{\partial}_\rho)\hat{x}^\nu \\ &= (\delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta))\hat{x}^\nu + \hat{x}^\mu([\hat{\partial}_\rho, \hat{x}^\nu] + \hat{x}^\nu \hat{\partial}_\rho) \\ &= (\delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta))\hat{x}^\nu + \hat{x}^\mu(\delta^\nu_\rho + f^\nu_\rho(\hat{\partial}, \theta) + \hat{x}^\nu \hat{\partial}_\rho), \\ \hat{\partial}_\rho \hat{x}^\nu \hat{x}^\mu &= \dots \\ &= (\delta^\nu_\rho + f^\nu_\rho(\hat{\partial}, \theta))\hat{x}^\mu + \hat{x}^\nu(\delta^\mu_\rho + f^\mu_\rho(\hat{\partial}, \theta) + \hat{x}^\mu \hat{\partial}_\rho), \end{aligned}$$

and

$$\hat{\partial}_\rho \theta^{\mu\nu} = \theta^{\mu\nu} \hat{\partial}_\rho.$$

Adding these three terms together we see that

$$\hat{\partial}_\rho \left([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}\right) = \left([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}\right) \hat{\partial}_\rho = 0 \tag{4.17}$$

is fulfilled for $f^\mu_\rho(\hat{\partial}, \theta) = 0$. Therefore,

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho. \tag{4.18}$$

There are no additional terms in (4.18). This is due to the fact that the right-hand side of (4.4) is constant. In the next chapter we study the κ-deformed space which is an example for a Lie algebra deformation. There it is not possible to set $f^\mu_\rho(\hat{\partial}, \kappa) = 0$ and additional terms arise.

In order to represent the derivative $\hat{\partial}_\rho$ on $\mathscr{A}^\star_x$ we use the following scheme:

$$\begin{tikzcd} \hat{f}(\hat{x}) \arrow[r, mapsto] \arrow[d, mapsto, "\hat{\partial}_\mu"'] & f(x) \arrow[d, mapsto, "\partial^\star_\mu"] \\ (\hat{\partial}_\mu \hat{f})(\hat{x}) \arrow[r, mapsto] & (\partial^\star_\mu f)(x)\,. \end{tikzcd} \tag{4.19}$$

First, the elements $\hat{f}(\hat{x})$ and $(\hat{\partial}_\mu \hat{f})(\hat{x})$ are mapped to $\mathscr{A}_x^\star$ by using the PBW property. Then comparing the images $f(x)$ and $(\partial_\mu^\star f)(x)$ the form of the operator $\partial_\mu^\star$ is deduced. In the case of θ-deformed space the representation of $\hat{\partial}_\rho$ on $\mathscr{A}_x^\star$ is given by the usual partial derivative

$$\hat{\partial}_\rho \mapsto \partial_\rho^\star = \partial_\rho . \tag{4.20}$$

This derivative has the undeformed Leibniz rule

$$\begin{aligned}\Big(\partial_\rho^\star(f\star g)\Big) = \partial_\rho(f\star g) &= \Big(\partial_\rho^\star f\Big)\star g + f\star\Big(\partial_\rho^\star g\Big)\\ &= \big(\partial_\rho f\big)\star g + f\star\big(\partial_\rho g\big) .\end{aligned} \tag{4.21}$$

Integral

To be able to write down actions we have to introduce an integral on this space. One can check that the usual integral on the commutative space is cyclic,[2] that is, it fulfills

$$\int \mathrm{d}^4x\ f\star g = \int \mathrm{d}^4x\ g\star f = \int \mathrm{d}^4x\ f\cdot g. \tag{4.22}$$

Note that from now on we work in four dimensions. However, all the results in this chapter can be generalized to higher dimensions as well. From (4.22) it follows

$$\int \mathrm{d}^4x\ (f_1\star f_2\star\ldots\star f_k) = \int \mathrm{d}^4x\ (f_k\star f_1\star\ldots\star f_{k-1}), \tag{4.23}$$

that is, cyclic permutations under the integral are allowed. This is important if we want to define the variational principle which is used to derive equations of motion, see Chap. 1. We use the usual Leibniz rule for the functional variation and then use cyclicity (4.23) to omit one $\star$ and extract the result. For example,

$$\begin{aligned}\frac{\delta}{\delta g(y)}\int \mathrm{d}^4x\ f\star g\star h &= \int \mathrm{d}^4x\ f\star\left(\frac{\delta}{\delta g(y)}g\right)\star h\\ &= \int \mathrm{d}^4x\ f\star\delta^{(4)}(y-x)\star h\\ &= \int \mathrm{d}^4x\ \delta^{(4)}(y-x)\star(h\star f)\\ &= \int \mathrm{d}^4x\ \delta^{(4)}(y-x)(h\star f) = h\star f\,(y).\end{aligned} \tag{4.24}$$

Here $\delta^{(4)}(y-x)$ is the usual four-dimensional commutative Dirac delta function.

[2] Note that (4.22) is fulfilled if suitable boundary conditions at infinity are chosen.

4.3 Twisted gauge theory

A non-abelian gauge group is generated by the hermitian generators T^a that fulfill

$$[T^a, T^b] = i f^{abc} T^c, \quad a = 1, \dots, n, \tag{4.25}$$

where f^{abc} are structure constants of the group and a sum over repeated indices is understood. From the Jacobi identities

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0, \tag{4.26}$$

it follows that

$$f^{abc} f^{dce} + f^{ace} f^{bdc} + f^{dac} f^{bce} = 0. \tag{4.27}$$

The matter field $\psi(x)$ is in a certain irreducible representation (fundamental for example) of this group. Under the undeformed gauge transformations it transforms as follows:

$$\psi(x) \to \psi'(x) = e^{i\alpha^a(x)T^a} \psi(x) \equiv U_\alpha(x)\psi(x), \tag{4.28}$$

or infinitesimally

$$\delta_\alpha \psi(x) = i\alpha^a(x) T^a \psi(x) \equiv i\alpha(x)\psi(x). \tag{4.29}$$

Note that the parameter of the above transformations is x-dependent, that is, transformations are local. Transformations (4.29) close in the algebra

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{-i[\alpha,\beta]}. \tag{4.30}$$

The Leibniz rule is given by

$$\begin{aligned} \delta_\alpha(\phi \cdot \psi) &= (\delta_\alpha \phi) \cdot \psi + \phi \cdot (\delta_\alpha \psi) \\ &= i\alpha^a \cdot \Big((T^a_\phi \phi) \cdot \psi + \phi \cdot (T^a_\psi \psi) \Big), \end{aligned} \tag{4.31}$$

where the generators T^a_ϕ and T^a_ψ are generators of the Lie algebra in the appropriate representation.

4.3.1 Gauge transformations

The deformed (twisted) gauge transformations [1, 2] were defined in Chap. 2 as follows:

$$\delta^\star_\alpha \psi = iX^\star_\alpha \star \psi = iX^\star_{\alpha^a} \star T^a \psi = i\alpha \cdot \psi, \tag{4.32}$$

where the operator $X^\star$ was introduced in Chap. 1 and is given by

$$X^\star_{\alpha^a} = \sum_{n=0}^{\infty} \left(-\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\rho_1\sigma_1} \dots \theta^{\rho_n\sigma_n}$$
$$\big(\partial_{\rho_1} \dots \partial_{\rho_n} \alpha^a(x)\big) \star \partial_{\sigma_1} \dots \partial_{\sigma_n}. \tag{4.33}$$

The operators $X^\star$ fulfill $X^\star_f \star X^\star_g = X^\star_{f\cdot g}$ as it was shown in Chap. 1. Therefore, we conclude that this transformations close in the algebra (4.30)

$$[\delta^\star_\alpha, \delta^\star_\beta]\psi = -i\delta^\star_{[\alpha,\beta]}\psi. \tag{4.34}$$

Now we have to learn how to transform products of fields. We start with defining the transformation law of $\star$-product of two fields as in the commutative case

$$\begin{aligned}\delta^\star_\alpha(\phi \star \psi) &= iX^\star_{\alpha^a} \star \{T^a_\phi \phi \star \psi + \phi \star T^a_\psi \psi\} \\ &= i\alpha^a \cdot \{T^a_\phi \phi \star \psi + \phi \star T^a_\psi \psi\}.\end{aligned} \tag{4.35}$$

In the last line the definition of $X^\star_\alpha$ given in (4.33) was used.

If we now take the usual Leibniz rule (4.31) and just insert $\star$-products everywhere we obtain

$$\begin{aligned}\delta^\star_\alpha(\phi \star \psi) &= (\delta^\star_\alpha \phi) \star \psi + \phi \star (\delta^\star_\alpha \psi) \\ &= i(\alpha^a T^a_\phi \phi) \star \psi + i\phi \star (\alpha^a T^a_\psi \psi).\end{aligned} \tag{4.36}$$

Clearly, the right-hand sides of (4.35) and (4.36) are not equal because the $\star$-product (4.12) is not commutative. Therefore, we formulate our deformed Leibniz rule in the following way:

$$\delta^\star_\alpha(\phi \star \psi) = (\delta^\star_\alpha \phi) \star \psi + \phi \star (\delta^\star_\alpha \psi) + \text{ additional terms}. \tag{4.37}$$

These additional terms are found by comparing the right-hand sides of the demanded transformation law (4.35) and (4.37). Expanding the $\star$-products up to first order in the deformation parameter θ and arranging terms leads to

$$\begin{aligned}\text{additional terms} &= \frac{1}{2}\theta^{\rho\sigma}\alpha^a \Big((T^a_\phi \partial_\rho \phi) \cdot (\partial_\sigma \psi) + \partial_\rho \phi \cdot (T^a_\psi \partial_\sigma \psi)\Big) + \mathcal{O}(\theta^2) \\ &= -\frac{i}{2}\theta^{\rho\sigma} \Big((\delta^\star_{\partial_\rho \alpha}\phi) \star (\partial_\sigma \psi) + (\partial_\rho \phi) \star (\delta^\star_{\partial_\sigma \alpha}\psi)\Big) + \mathcal{O}(\theta^2),\end{aligned}$$

which gives the following Leibniz rule:

$$\begin{aligned}\delta^\star_\alpha(\phi \star \psi) &= (\delta^\star_\alpha \phi) \star \psi + \phi \star (\delta^\star_\alpha \psi) \\ &-\frac{i}{2}\theta^{\rho\sigma} \Big((\delta^\star_{\partial_\rho \alpha}\phi) \star (\partial_\sigma \psi) + (\partial_\rho \phi) \star (\delta^\star_{\partial_\sigma \alpha}\psi)\Big) + \mathcal{O}(\theta^2).\end{aligned} \tag{4.38}$$

One can continue like this to second and higher orders, see Chap. 1 for details. The full result for the deformed Leibniz rule is finally

$$\delta_\alpha^\star(\phi\star\psi) = (\delta_\alpha^\star\phi)\star\psi+\phi\star(\delta_\alpha^\star\psi)+\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\right)^n\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n} \quad (4.39)$$
$$\Big((\delta_{\partial_{\rho_1}\dots\partial_{\rho_n}\alpha}\phi)\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)+(\partial_{\rho_1}\dots\partial_{\rho_n}\phi)\star(\delta_{\partial_{\sigma_1}\dots\partial_{\sigma_n}\alpha}\psi)\Big).$$

Using the Hopf algebra language, this deformed Leibniz rule comes from the deformed coproduct of $\delta_\alpha^\star$ transformations. The deformed coproduct is obtained by applying the twist operator on the undeformed Hopf algebra of gauge transformations δ_α. In that way the algebra itself remains unchanged, but the comultiplication (which leads to the Leibniz rule) changes. The Hopf algebra techniques are discussed in Chap. 2 and will also be subjects of Chaps. 7 and 8 so we do not go into details here.

4.3.2 Field strength tensor

Having found the Leibniz rule for $\delta_\alpha^\star$ transformations, we proceed like in the commutative case. First, covariant derivative is introduced as

$$\begin{aligned} D_\mu^\star\psi &= \partial_\mu\psi - iA_\mu\star\psi, \\ \delta_\alpha^\star(D_\mu^\star\psi) &= iX_{\alpha^a}^\star\star T^a(D_\mu^\star\psi) = i\alpha(D_\mu^\star\psi), \end{aligned} \quad (4.40)$$

where A_μ is the noncommutative gauge field. Using (4.39) when explicitly calculating (4.40) gives

$$\begin{aligned} \delta_\alpha^\star A_\mu &= \partial_\mu\alpha + iX_{\alpha^a}^\star\star[T^a,A_\mu] \\ &= \partial_\mu\alpha + i[\alpha,A_\mu]. \end{aligned} \quad (4.41)$$

From this transformation law it follows that A_μ can be taken to be Lie algebra valued, $A_\mu = A_\mu^a T^a$.

However, introducing the field strength tensor $F_{\mu\nu}$ as

$$F_{\mu\nu}\star\psi = i[D_\mu^\star \stackrel{\star}{,} D_\nu^\star]\psi \quad (4.42)$$

gives

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu], \\ \delta_\alpha^\star F_{\mu\nu} &= iX_{\alpha^a}^\star\star[T^a,F_{\mu\nu}] = i[\alpha,F_{\mu\nu}]. \end{aligned} \quad (4.43)$$

If we assume that $A_\mu = A_\mu^a T^a$, that is, A_μ is Lie algebra valued, for $F_{\mu\nu}$ we obtain

$$\begin{aligned} F_{\mu\nu} &= F_{1\mu\nu}^a T^a + F_{2\mu\nu}^{ab}\frac{1}{2}\{T^a,T^b\} = F_{1\mu\nu}+F_{2\mu\nu}, \\ F_{1\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \frac{1}{2}f^{abc}\{A_\mu^b \stackrel{\star}{,} A_\nu^c\}, \\ F_{2\mu\nu}^{ab} &= -i[A_\mu^a \stackrel{\star}{,} A_\nu^b], \end{aligned} \quad (4.44)$$

where $\{T^a, T^b\} = T^aT^b + T^bT^a$ and $\{A^b_\mu \stackrel{\star}{,} A^c_\nu\} = A^b_\mu \star A^c_\nu + A^c_\nu \star A^b_\mu$. The anticommutator of generators $\{T^a, T^b\}$ is not Lie algebra valued in general. In the special case of $U(N)$ gauge group one can express the anticommutator $\{T^a, T^b\}$ in the generators T^a only (no products of generators) [3]. In the case of $SU(N)$ groups this is not possible. Still, we are interested in $SU(N)$ gauge theories because of their importance for the formulation of the standard model. Therefore, $F_{\mu\nu}$ (4.44) will not be Lie algebra valued because of the term $F_{2\mu\nu} = F^{ab}_{2\mu\nu}\frac{1}{2}\{T^a, T^b\}$. The good news is that both $F_{1\mu\nu}$ and $F_{2\mu\nu}$ transform covariantly

$$\begin{aligned}\delta^\star_\alpha F_{1\mu\nu} &= iX^\star_{\alpha^a} \star [T^a, F_{1\mu\nu}] = i[\alpha, F_{1\mu\nu}],\\ \delta^\star_\alpha F_{2\mu\nu} &= iX^\star_{\alpha^a} \star [T^a, F_{2\mu\nu}] = i[\alpha, F_{2\mu\nu}].\end{aligned} \tag{4.45}$$

If we want to stay in the Lie algebra we could take just $F^a_{1\mu\nu}$ part of the full $F_{\mu\nu}$ and formulate the action with it only. Note that $F^a_{\mu\nu}$ also has the good classical limit, when $\theta \to 0$ it reduces to the commutative $F_{\mu\nu}$. One can also include matter fields and formulate "gauge + matter" action, derive equations of motion, analyze their solutions, and so on. Unfortunately, some problems arise in this procedure. To see clearly what is causing them we now analyze only the gauge part of the action.

4.3.3 Equations of motion

Let us write the action for the gauge field as

$$S_{\text{gauge}} = -\frac{1}{4}\int \mathrm{d}^4x\, F^a_{1\mu\nu} \star F_1^{\mu\nu\,a}, \tag{4.46}$$

that is, taking only the Lie algebra-valued part of the full $F_{\mu\nu}$. Using the variational principle (4.24) gives

$$\frac{\delta S_{\text{gauge}}}{\delta A^a_\rho} = \partial_\mu F_1^{\mu\rho\,a} + \frac{1}{2} f^{abc}\{A^b_\mu \stackrel{\star}{,} F_1^{\mu\rho\,c}\} = 0. \tag{4.47}$$

From the antisymmetry of $F^a_{\mu\nu}$ it follows that $\partial_\rho\partial_\mu F^{\mu\rho^a} = 0$ and this leads to

$$\partial_\rho \left(\frac{1}{2} f^{abc}\{A^b_\mu \stackrel{\star}{,} F_1^{\mu\rho\,c}\}\right) = \partial_\rho J^{\rho\,a} = 0, \tag{4.48}$$

where the conserved current is defined as $J^{\rho\,a} = \frac{1}{2} f^{abc}\{A^b_\mu \stackrel{\star}{,} F_1^{\mu\rho\,c}\}$. However, when we check its conservation explicitly using the equations of motion (4.47) we obtain

$$\begin{aligned}\partial_\rho J^{\rho\,a} &= \dots\\ &= \frac{1}{4}\Big(f^{abc} f^{bdh} A^h_\mu \star F_1^{\mu\rho\,c} \star A^d_\rho\\ &\quad + f^{abd} f^{bhc} F_1^{\mu\rho\,c} \star A^d_\rho \star A^h_\mu + f^{abh} f^{bcd} A^d_\rho \star A^h_\mu \star F_1^{\mu\rho\,c}\Big) \neq 0.\end{aligned} \tag{4.49}$$

The final result was obtained by using the Jacobi identity (4.27). This tells us that $J^{\rho\, a}$ is not conserved on the equations of motion, that is, equations (4.47) are not consistent. This signalizes that we are doing something wrong. It comes out that the problem arose when we took only the Lie algebra-valued part of $F_{\mu\nu}$ and not the full $F_{\mu\nu}$.

Let us now take the full $F_{\mu\nu}$ (4.43) and assume that A_μ and $F_{\mu\nu}$ are $n\times n$ matrix valued where n is the dimension of the Lie algebra representation. We obtain

$$\begin{aligned} S_{\text{gauge}} &= -\frac{1}{4}\int \mathrm{d}^4x \,\mathrm{Tr}(F_{\mu\nu}\star F^{\mu\nu}) \\ &= -\frac{1}{4}\int \mathrm{d}^4x \,(F_{\mu\nu})_{\text{AB}}\star (F^{\mu\nu})_{\text{BA}}, \end{aligned} \tag{4.50}$$

where in the last line matrix indices A and B are written explicitly. The equations of motion are given by

$$\frac{\delta S_{\text{gauge}}}{\delta (A_\rho)_{\text{AB}}} = \left(\partial_\mu F^{\mu\rho}\right)_{\text{AB}} - i\left([A_\mu \stackrel{\star}{,} F^{\mu\rho}]\right)_{\text{AB}} = 0. \tag{4.51}$$

Again, using the antisymmetry of $F_{\mu\nu}$, we obtain

$$\partial_\rho \left(i[A_\mu \stackrel{\star}{,} F^{\mu\rho}]\right) = \partial_\rho J^\rho = 0. \tag{4.52}$$

This time the conserved current is defined as $J^\rho = i[A_\mu \stackrel{\star}{,} F^{\mu\rho}]$, compare with Chap. 2, Sect. 2.4. The result (4.52) can be checked explicitly by using the equations of motion (4.51) and the Jacobi identity. After differentiation,

$$\partial_\rho [A_\mu \stackrel{\star}{,} F^{\mu\rho}] = [\partial_\rho A_\mu \stackrel{\star}{,} F^{\mu\rho}] + [A_\mu \stackrel{\star}{,} \partial_\rho F^{\mu\rho}], \tag{4.53}$$

we antisymmetrize $\partial_\rho A_\mu$ because $F_{\mu\rho}$ is antisymmetric in μ and ρ. Then we use

$$\frac{1}{2}(\partial_\rho A_\mu - \partial_\mu A_\rho) = \frac{i}{2}F_{\rho\mu} + \frac{i}{2}[A_\rho \stackrel{\star}{,} A_\mu] \tag{4.54}$$

and insert it into (4.53). The commutator $[F^{\mu\rho} \stackrel{\star}{,} F_{\mu\rho}]$ vanishes and only $\frac{i}{2}[[A_\rho \stackrel{\star}{,} A_\mu] \stackrel{\star}{,} F^{\mu\rho}]$ remains from the first term in (4.53). For the second term in (4.53) we use the equations of motion (4.51). Finally, all terms that are left add up to zero if we use the Jacobi identity, so J^ρ is conserved on the equations of motion.

To conclude, the consistency of equations of motion forces us to take $F_{\mu\nu}$ in the form (4.43), that is non-Lie algebra valued. Then there is no reason to assume that A_μ itself is Lie algebra valued. The consistent equations of motion follow if A_μ and $F_{\mu\nu}$ are taken to be $n\times n$ matrix valued in the n-dimensional representation of the Lie algebra. But we can also take the gauge field A_μ to be enveloping algebra valued

$$A_\mu = A_\mu^a T^a + A_\mu^{ab}\frac{1}{2}\{T^a, T^b\} + \cdots. \tag{4.55}$$

Then the field strength tensor $F_{\mu\nu}$ (4.43) is also enveloping algebra valued and from the equations of motion (4.51) it follows that A_μ and $F_{\mu\nu}$ will remain enveloping algebra valued.

Looking at (4.55) we see that in this way we are introducing extra degrees of freedom via fields A_μ^{ab}, ... In general, there will be infinitely many new fields, these need a physical interpretation. A possible solution might be to generate large masses for these fields via some form of Higgs mechanism. In this way they become unobservable for today's experiments. Nevertheless this still remains an open question and needs to be analyzed in future. Let us mention that the special example of $SU(2)$ gauge transformations in the two-dimensional representation was discussed in [1, 2]. There it was shown that in order to have consistent equations of motion one has to enlarge $SU(2)$ by adding only one new abelian field, that is, to replace $SU(2)$ by $U(2)$. This will be also true for other $SU(N)$ gauge groups in the N-dimensional representation. It is enough to enlarge the gauge group from $SU(N)$ to $U(N)$ since for $U(N)$ in the N-dimensional representation the $\star$-commutator $[\ \stackrel{\star}{,}\ ]$ closes in the algebra.

Another important result is (4.52). Almost for free we got a conserved current for the deformed symmetry. Note that in the case of deformed symmetry one cannot apply the usual Noether's theorem to derive conserved quantities. Some work in this context is done in [22–24], but this problem still remains a subject of further research.

4.4 Seiberg–Witten gauge theory

In this section we describe a different approach to gauge theories on deformed spaces. The infinitesimal noncommutative gauge transformations are now defined as [3–5]

$$\delta_\Lambda^{\mathrm{sw}}\psi(x) = i\Lambda \star \psi(x), \tag{4.56}$$

where Λ is the noncommutative gauge parameter and ψ is the noncommutative matter field. Note that $\Lambda = \Lambda(x)$ is a function and not a differential operator like in (4.32). Before proceeding to the standard construction of a covariant derivative one should check if these transformations close in the algebra (4.30). If the noncommutative gauge parameter Λ is supposed to be Lie algebra valued $\Lambda(x) = \Lambda^a(x)T^a$ an explicit calculation gives

$$\begin{aligned}
&(\delta_{\Lambda_1}^{\mathrm{sw}}\delta_{\Lambda_2}^{\mathrm{sw}} - \delta_{\Lambda_2}^{\mathrm{sw}}\delta_{\Lambda_1}^{\mathrm{sw}})\psi = (\Lambda_1\star\Lambda_2 - \Lambda_2\star\Lambda_1)\star\psi \\
&= \frac{1}{2}\Big([\Lambda_1^a \stackrel{\star}{,} \Lambda_2^b]\{T^a, T^b\} + \{\Lambda_1^a \stackrel{\star}{,} \Lambda_2^b\}[T^a, T^b]\Big)\star\psi.
\end{aligned} \tag{4.57}$$

Note that now $\delta_\Lambda^{\mathrm{sw}}$ has the undeformed Leibniz rule

$$\delta_\Lambda^{\mathrm{sw}}(\phi\star\psi) = (\delta_\Lambda^{\mathrm{sw}}\phi)\star\psi + \phi\star(\delta_\Lambda^{\mathrm{sw}}\psi).$$

The left-hand side of (4.57) in general does not close in the Lie algebra because of the first term in the last line. Namely, an anticommutator of two generators is in general no longer in the Lie algebra of generators. There are two ways of solving this problem. One is to consider only $U(N)$ gauge theories (and with some difficulty $SO(N)$ and $Sp(N)$ gauge theories [7, 8]) since then the anticommutator of generators is still in the Lie algebra of generators. This is the approach taken in [3] where the Seiberg–Witten map was constructed for the first time. It enables the study of non-expanded (in orders of the deformation parameter) noncommutative field theories. Actions describing these field theories look the same as the actions describing corresponding undeformed theories, except that instead of the usual pointwise multiplication, the Moyal $\star$-product (4.12) is used to multiply fields. Quantization of the non-expanded theories leads to the mixing of ultraviolet (UV) and infrared (IR) divergences which is known in the literature as UV/IR mixing [25, 26]. However, there are other approaches which give different results. For example, a more careful treatment of the time ordering and the perturbation theory results in an UV finite S-matrix and no UV/IR mixing occurs, see [27, 28] and references therein. Yet another approach is used in [29, 30]. There an additional term is added to the action for ϕ^4 scalar field theory on the θ-deformed space. The new term is nonlocal and it explicitly breaks the translational invariance; however, it renders the new action renormalizable to all orders.

The non-expanded NC field theories we will not consider here. Still, it is interesting to make a few more remarks before we proceed further.

In [31] a noncommutative quantum electrodynamics was discussed by using a non-expanded approach. The same action as the action (4.50) in the case of $U(1)$ gauge theory was introduced and some quantum properties, explicitly UV/IR mixing, were discussed. Note that these actions although they look the same are invariant under different symmetry. The action (4.50) is invariant under twisted gauge transformations, while the action discussed in [31] is invariant under (4.56) transformations. Also note that A_μ in the approach of [31] is just the commutative $U(1)$ gauge field. Some non-perturbative solutions of equations of motion for non-expanded noncommutative gauge theories were studied in [32–34]. The equations of motion have the same form as (4.51) but are covariant with respect to (4.56) transformations, while (4.51) are covariant with respect to twisted gauge transformations so the interpretation is different.

The approach we will follow here is the enveloping algebra approach [4, 5]. Namely, the enveloping algebra is big enough and the transformations (4.56) will close in it.

4.4.1 Enveloping algebra approach

To start with, we define the basis in the enveloping algebra (we choose symmetric ordering)

$$\begin{aligned}
:T^a: &= T^a,\\
:T^aT^b: &= \frac{1}{2}(T^aT^b + T^bT^a),\\
&\dots,\\
:T^{a_1}\dots T^{a_l}: &= \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\dots T^{\sigma(a_l)}).
\end{aligned}$$

The gauge parameter Λ is enveloping algebra valued

$$\begin{aligned}
\Lambda(x) &= \sum_{l=0}^{\infty}\sum_{\text{basis}}\Lambda^{l\,a_1\dots a_l}(x) :T^{a_1}\dots T^{a_l}:\\
&= \Lambda^{0\,a}(x):T^a: + \Lambda^{1\,a_1a_2}(x):T^{a_1}T^{a_2}: + \cdots.
\end{aligned} \tag{4.58}$$

In this case (4.57) closes since we work in the enveloping algebra. Now one defines a covariant derivative

$$D^{\star}_{\mu}\psi(x) = \partial_\mu\psi(x) - iV_\mu \star \psi(x), \tag{4.59}$$

where V_μ is a noncommutative gauge field. The transformation law of the covariant derivative is given by

$$\delta^{\mathrm{sw}}_{\Lambda}(D^{\star}_{\mu}\psi(x)) = i\Lambda \star D^{\star}_{\mu}\psi(x). \tag{4.60}$$

Then the transformation law for the noncommutative gauge field follows from (4.60)

$$\delta^{\mathrm{sw}}_{\Lambda}V_\mu = \partial_\mu\Lambda + i[\Lambda \stackrel{\star}{,} V_\mu]. \tag{4.61}$$

From here it is obvious that the gauge field V_μ has to be enveloping algebra as well

$$V_\mu = \sum_{l=0}^{\infty}\sum_{\text{basis}}V_\mu^{l\,a_1\dots a_l} :T^{a_1}\dots T^{a_l}: .$$

It looks as if we obtained a theory with infinitely many degrees of freedom. Fortunately there is a solution to this problem and it is given in terms of the Seiberg–Witten (SW) map [3] .

4.4.2 Seiberg–Witten map

The basic idea behind this map is to suppose that all higher order degrees of freedom depend only on the degrees of freedom present at zeroth order, the Lie algebra-valued gauge parameter $\Lambda^{0\,a}T^a$ and the Lie algebra-valued gauge field $V_\mu^{0\,a}T^a$. If such a reduction of degrees of freedom is possible, it means that the gauge theory on a NC space can be related to and is determined entirely by the gauge field theory on commutative space. Especially, the number of degrees of freedom of the NC gauge theory and the gauge theory on commutative space would be equal in this case.

Studying the cohomology of the enveloping algebra-valued gauge theory one can show that this reduction is indeed possible [6]. Here we perform explicit calculations up to first order in the deformation parameter θ and do not go into details concerning the existence of such a construction to all orders in θ.

Let us suppose that the noncommutative gauge parameter Λ depends on the commutative gauge parameter $\alpha = \alpha^a T^a$ and the commutative gauge field $A_\mu = A^a_\mu T^a$ and their derivatives, that is,

$$\Lambda \equiv \Lambda_\alpha = \Lambda_\alpha(x; \alpha, A_\mu). \tag{4.62}$$

Then the gauge transformation $\delta^{\mathrm{sw}}_\Lambda$ can be related to the commutative gauge transformation δ_α of the expanded Λ_α (4.58). Therefore, we have

$$\delta^{\mathrm{sw}}_\Lambda = \delta^{\mathrm{sw}}_{\Lambda_\alpha} \equiv \delta^{\mathrm{sw}}_\alpha, \tag{4.63}$$

where the last definition is introduced in order to avoid the double index notation. One should notice that $\delta_\alpha \Lambda_\beta \neq 0$ because Λ_β depends on the commutative gauge field A_μ as well and

$$\delta_\alpha A_\mu = \partial_\mu \alpha - i[A_\mu, \alpha]. \tag{4.64}$$

Inserting $\Lambda_\alpha = \Lambda_\alpha(x; \alpha, A_\mu)$ into (4.57) gives

$$i(\delta^{\mathrm{sw}}_\alpha \Lambda_\beta - \delta^{\mathrm{sw}}_\beta \Lambda_\alpha) \star \psi + (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi = i\Lambda_{-i[\alpha,\beta]} \star \psi. \tag{4.65}$$

Since this has to be valid for any matter field ψ, we obtain

$$\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha + i(\delta^{\mathrm{sw}}_\alpha \Lambda_\beta - \delta^{\mathrm{sw}}_\beta \Lambda_\alpha) = \delta^{\mathrm{sw}}_{-i[\alpha,\beta]}. \tag{4.66}$$

Equation (4.66) can be solved perturbatively. Therefore, one has to expand the $\star$-product and also expand Λ_α in orders of the deformation parameter[3] θ as

$$\Lambda_\alpha = \Lambda^0_\alpha + \Lambda^1_\alpha + \cdots + \Lambda^k_\alpha + \cdots. \tag{4.67}$$

Here Λ^0_α is of the zeroth order in θ, Λ^1_α is of the first order in θ, and so on. In the zeroth order the noncommutative gauge parameter is just the gauge parameter of the undeformed (commutative) gauge theory, $\Lambda^0_\alpha = \alpha = \alpha^a T^a$. Then Λ^1_α is the solution of the inhomogeneous equation

$$\begin{aligned}\delta^{\mathrm{sw}}_\alpha \Lambda^1_\beta - \delta^{\mathrm{sw}}_\beta \Lambda^1_\alpha \; - \; i[\alpha, \Lambda^1_\beta] - i[\Lambda^1_\alpha, \beta] - \Lambda^1_{-i[\alpha,\beta]}& \\ = -\frac{1}{2}\theta^{\mu\nu}\{\partial_\mu \alpha, \partial_\nu \beta\}.&\end{aligned} \tag{4.68}$$

The solution, up to first order in θ, is given by

$$\Lambda_\alpha = \alpha - \frac{1}{4}\theta^{\mu\nu}\{A_\mu, \partial_\nu \alpha\}. \tag{4.69}$$

[3] Note that this expansion is different than the one in (4.58). This difference is clearly visible in the solutions for the second and higher orders of the expansion (4.67), see [35] for details and Sect. 5.5 for an explicit example.

Note that this solution is not unique, one can always add solutions of the homogeneous equation

$$\delta_\alpha^{\mathrm{sw}} \Lambda_\beta^1 - \delta_\beta^{\mathrm{sw}} \Lambda_\alpha^1 - i[\alpha, \Lambda_\beta^1] - i[\Lambda_\alpha^1, \beta] - \Lambda^1_{-i[\alpha,\beta]} = 0 \tag{4.70}$$

to it, see for example [4, 5]. One can add $c\Lambda_\alpha^{1\mathrm{hom}}$ to the solution (4.69), but since the nonuniqueness of Seiberg–Witten map is not our main topic here, we choose $c = 0$. The same (omitting the solutions of the homogeneous equations) we do in the rest of this chapter. Detailed analysis of nonuniqueness of Seiberg–Witten map can be found in [35].

Now one can solve the SW map for the matter field ψ using (4.56) and (4.69). Matter field is also expanded in orders of the deformation parameter θ as $\psi = \psi^0 + \psi^1 + \cdots$ and ψ^0 is the matter field in the undeformed gauge theory with $\delta_\alpha \psi^0 = i\alpha\psi^0$. Inserting this into (4.56) leads to equations for ψ^1 and higher order terms which can be solved. The equation for ψ^1 reads

$$\delta_\alpha \psi^1 = i\alpha\psi^1 + i\Lambda_\alpha^1 \psi^0 - \frac{1}{2}\theta^{\rho\sigma}(\partial_\rho \alpha)(\partial_\sigma \psi^0). \tag{4.71}$$

Using the solution for the noncommutative gauge parameter (4.69) we find (up to first order in θ again)

$$\psi = \psi^0 - \frac{1}{2}\theta^{\mu\nu} A_\mu (\partial_\nu \psi^0) + \frac{i}{8}\theta^{\mu\nu}[A_\mu, A_\nu]\psi^0. \tag{4.72}$$

In the same way the gauge field V_μ is expanded in orders of the deformation parameter θ as

$$\begin{aligned} V_\mu &= \sum_{l=0}^{\infty} \sum_{\text{basis}} V_\mu^{l\, a_1 \ldots a_l} : T^{a_1} \ldots T^{a_l} : . \\ &= V_\mu^{0a} T^a + V_\mu^{1ab} : T^a T^b : + \cdots . \end{aligned} \tag{4.73}$$

Inserting these expansions into the transformation law (4.61) the following equations are obtained

$$\theta^0 : \quad \delta_\alpha V_\mu^0 = \partial_\mu \alpha + i[\alpha, V_\mu^0], \tag{4.74}$$

$$\begin{aligned} \theta^1 : \quad \delta_\alpha V_\mu^1 &= \partial_\mu \Lambda^1 + i[\alpha, V_\mu^1] + i[\Lambda^1, V_\mu^0] \\ &\quad + i[\alpha \stackrel{\star^1}{,} V_\mu^0]. \end{aligned} \tag{4.75}$$

The commutator $[\alpha \stackrel{\star^1}{,} V_\mu^0] = i\theta^{\rho\sigma}(\partial_\rho \alpha)(\partial_\sigma V_\mu^0)$ stands for the first order in θ of the full $\star$-commutator $[\alpha \stackrel{\star}{,} V_\mu^0]$. The zeroth-order equation is consistent with the assumption that V_μ^0 is the commutative gauge field $V_\mu^0 = A_\mu = A_\mu^a T^a$. Using the solution for Λ_α (4.69), the first-order solution of (4.75) is given by

$$V_\mu^1 = \frac{1}{4}\theta^{\rho\sigma}\left(\{F_{\rho\mu}, A_\sigma\} - \{A_\rho, \partial_\sigma A_\mu\}\right). \tag{4.76}$$

That this is really a solution of (4.75) one can check explicitly by using the transformation law (4.64). Finally, the solution for the noncommutative gauge field V_μ up to first order in θ reads

$$V_\rho = A_\rho + \frac{1}{4}\theta^{\mu\nu}\left(\{F_{\mu\rho}, A_\nu\} - \{A_\mu, \partial_\nu A_\rho\}\right). \tag{4.77}$$

The field strength tensor $F^\star_{\mu\nu}$ is calculated from

$$\begin{aligned} F^\star_{\mu\nu} &= i[D^\star_\mu \stackrel{\star}{,} D^\star_\nu] \\ &= \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu \stackrel{\star}{,} V_\nu] \\ &= F_{\mu\nu} + \frac{1}{4}\theta^{\rho\sigma}\left(2\{F_{\rho\mu}, F_{\sigma\nu}\} + \{D_\rho F_{\mu\nu}, A_\sigma\} - \{A_\rho, \partial_\sigma F_{\mu\nu}\}\right). \end{aligned} \tag{4.78}$$

In solutions (4.76), (4.77), and (4.78) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the commutative field strength tensor and its covariant derivative is given by $D_\rho F_{\mu\nu} = \partial_\rho F_{\mu\nu} - i[A_\rho, F_{\mu\nu}]$.

With the solutions of the SW map (4.69), (4.72), (4.77), and (4.78) we have enough information to write down the action for the Seiberg–Witten gauge theory. As an example, the action for spinor matter field is given by

$$S = \int \mathrm{d}^4x\ \bar{\psi}\star(i\gamma^\mu D^\star_\mu \psi - m\psi) - \frac{1}{4}\int \mathrm{d}^4x\ \mathrm{Tr}\ \left(F^{\star\,\mu\nu} \star F^\star_{\mu\nu}\right). \tag{4.79}$$

This action can be expanded in orders of the deformation parameter and analyzed perturbatively. One sees that no new fields appear (unlike in the “twisted” approach), but only new interactions and the deformation parameter enters as a coupling constant. Using this approach a formulation of a NC standard model has been done in [9, 10]. Then, since new interactions appear, some of the processes that are forbidden in the commutative standard model become allowed and can be analyzed [11, 12]. Also, renormalization of these theories has been discussed, see [36–39] and references therein.

Finally, let us remark that in the noncommutative case (where partial derivatives are frequently given by the commutator with coordinates) besides covariant derivatives one has the concept of covariant coordinates [40]. Using the Kontsevich formality maps [41] in order to construct covariant coordinates, the SW map to all orders for an arbitrary Poisson manifold was constructed for the NC gauge theories in [42, 43].

4.5 Comments

We presented two ways of introducing gauge theories on deformed spaces. Seiberg–Witten approach enables us to analyze some properties of gauge theories on NC spaces in a perturbative way, starting from the commutative gauge theory in the zeroth order. Unlike the “twisted” approach, it does not lead to new fields, but only to new interactions. On the other hand, in the "twisted" approach we know how to formulate deformed gauge symmetry in a well-defined mathematical language, that is, in terms of Hopf algebras. This tells us that we have really changed the commutative gauge symmetry in a well-defined way. To summarize we write explicitly the Hopf algebras of both "twisted" and Seiberg–Witten gauge transformations:

Twisted gauge transformations

$$\delta^\star_\alpha\delta^\star_\beta - \delta^\star_\beta\delta^\star_\alpha = \delta^\star_{-i[\alpha,\beta]}, \quad \delta^\star_\alpha\psi = i\alpha\psi, \tag{4.80}$$

$$\begin{aligned}\Delta\delta^\star_\alpha &= e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}(\delta^\star_\alpha\otimes 1 + 1\otimes\delta^\star_\alpha)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} \qquad (4.81)\\ &= \delta^\star_\alpha\otimes 1 + 1\otimes\delta^\star_\alpha - \frac{i}{2}\theta^{\rho\sigma}(\delta^\star_{(\partial^\star_\rho\alpha)}\otimes\partial_\sigma + \partial_\rho\otimes\delta^\star_{(\partial_\sigma\alpha)}) + \cdots,\end{aligned}$$

$$\varepsilon(\delta^\star_\alpha) = 0, \quad S(\delta^\star_\alpha) = -\delta^\star_\alpha. \tag{4.82}$$

Seiberg–Witten gauge transformations

$$\delta^{\mathrm{sw}}_\alpha\delta^{\mathrm{sw}}_\beta - \delta^{\mathrm{sw}}_\beta\delta^{\mathrm{sw}}_\alpha = \delta^{\mathrm{sw}}_{-i[\alpha,\beta]}, \quad \delta^{\mathrm{sw}}_\alpha\psi = i\Lambda_\alpha\star\psi, \tag{4.83}$$

$$\Delta\delta^{\mathrm{sw}}_\alpha = \delta^{\mathrm{sw}}_\alpha\otimes 1 + 1\otimes\delta^{\mathrm{sw}}_\alpha, \tag{4.84}$$

$$\varepsilon(\delta^{\mathrm{sw}}_\alpha) = 0, \quad S(\delta^{\mathrm{sw}}_\alpha) = -\delta^{\mathrm{sw}}_\alpha. \tag{4.85}$$

We see that in the case of Seiberg–Witten gauge transformations, the Hopf algebra is the same as in the case of undeformed gauge transformations, but the way fields transform is different. In the case of "twisted" gauge transformations the deformation is present in the coalgebra sector, namely in the deformed coproduct. This deformation then allows the introduction of a tensor calculus and the construction of gauge-invariant actions. An attempt to relate these two approaches is done in [44], but the connection between them still remains to be fully understood. The question which of them (if any) is more applicable to our real world also remains open and a subject of further research.

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Chapter 5
Another Example of Noncommutative Spaces: κ-Deformed Space

Marija Dimitrijević

In this chapter we discuss another type of noncommutative space, the κ-deformed space. It is an example of Lie algebra type of deformation of the usual commutative space. In the first part derivatives and the symmetry of this space are discussed. We start with the abstract algebra of operators and using the $\star$-product approach represent everything on the space of commuting coordinates. In the second part we describe how to construct noncommutative gauge theory on this space using the Seiberg–Witten approach.

5.1 Introduction

Gauge and gravity theories on noncommutative spaces were subject of previous chapters. Although only the θ-deformed (canonically deformed) space was discussed, the analysis was general enough so that it could be applied to other types of noncommutative spaces. In this chapter we want to consider a Lie algebra type of deformation, the κ-Minkowski space. We follow the outlined approach when representing the algebra of noncommuting coordinates and operators acting on it on the space of commuting coordinates. However, note that the deformed Poincaré algebra which we consider here does not follow from a twist. More details about twists and Hopf algebras will be given in the second part of the book, see Chaps. 7 and 8.

Historically, the κ-Minkowski space was first obtained in [1, 2] contracting the q-anti-de Sitter Hopf algebra $SO_q(3,2)$. The κ-Poincaré algebra was introduced in [3] as a dual symmetry structure to the κ-Poincaré group. Then the κ-Minkowski space is introduced as a module of this algebra. One of the reasons why the κ-Minkowski space has been studied in last years is that there is a quantum group symmetry acting on it. It has been believed until recently that a quantum group symmetry for the θ-deformed space does not exist (it was shown in [4–6] that that is not the case, however; see also Chap. 7, Sect. 7.7). Therefore, the κ-Minkowski space has been considered to be one of the simplest examples of possible deformations of

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the usual Minkowski space with a deformed symmetry (quantum group symmetry) acting on it. It is the so-called κ-Poincaré group. This space plays also an important role in the Doubly Special Relativity (DSR) theories [7–9]. These theories are introduced as a possible generalization of the Special Relativity. They are characterized by two invariants, the speed of light in vacuum c and the minimal length, Planck length l_P. For reviews on DSR see [10, 11].

We start from an abstract algebra of noncommuting coordinates. Then derivatives and Lorentz generators are introduced as operators which map this algebra to itself. By using the $\star$-product approach the abstract noncommutative space and the operators acting on it are mapped to the space of commuting coordinates. Finally, non-abelian gauge theories are formulated following the Seiberg–Witten approach as introduced in the previous chapter.

5.2 κ-deformed space

The κ-deformed space is a special example of the Lie algebra type of deformation. It is the abstract algebra $\hat{\mathscr{A}}_{\hat{x}}$ generated by coordinates $\hat{x}^\mu$ which fulfill

$$[\hat{x}^\mu,\hat{x}^\nu] = iC^{\mu\nu}_\rho \hat{x}^\rho, \tag{5.1}$$

where

$$C^{\mu\nu}_\rho = a(\delta^\mu_n \delta^\nu_\rho - \delta^\nu_n \delta^\mu_\rho), \quad \mu = 0,\dots,n. \tag{5.2}$$

The constant deformation vector a^μ of length a points in the n-th space-like direction, $a^n = a$. The parameter a is related to the frequently used parameter κ as $a = 1/\kappa$. Written more explicitly (5.1) reads

$$[\hat{x}^n,\hat{x}^l] = ia\hat{x}^l, \quad [\hat{x}^k,\hat{x}^l] = 0; \quad k,l = 0,1,\dots,n-1. \tag{5.3}$$

Latin indices denote the undeformed dimensions, n denotes the deformed dimension (deformed in the sense that $\hat{x}^n$ does not commute with the other coordinates), and Greek indices refer to all $n+1$ dimensions. Note that in the κ-Minkowski space of [1, 2, 12] time direction is noncommutative, while here we choose one space direction to be noncommutative.

As in the previous chapter derivatives are introduced as maps on the deformed coordinate space [13, 14]

$$\hat{\partial} : \hat{\mathscr{A}}_{\hat{x}} \to \hat{\mathscr{A}}_{\hat{x}}.$$

Such a map in particular has to map the ideal generated by the commutation relations of coordinates (5.2) into itself, that is,

$$\hat{\partial}_\rho \left([\hat{x}^\mu,\hat{x}^\nu] - iC^{\mu\nu}_\lambda \hat{x}^\lambda\right) = \left([\hat{x}^\mu,\hat{x}^\nu] - iC^{\mu\nu}_\lambda \hat{x}^\lambda\right)\hat{\partial}_\rho = 0. \tag{5.4}$$

If this is the case we say that the map $\hat{\partial}$ respects the commutation relations (5.2) or is consistent with them.

To find a suitable map we use the same technique as in the previous chapter. We make a general ansatz for the commutator of a derivative with a coordinate:

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu + \sum_j A_\mu^{\nu\rho_1\ldots\rho_j} \hat{\partial}_{\rho_1} \ldots \hat{\partial}_{\rho_j}. \tag{5.5}$$

We mention again that $[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0$ in our approach. The coefficient functions $A_\mu^{\nu\rho_1\ldots\rho_j}$ are constant and vanish in the commutative limit, $a \to 0$. Requiring consistency of (5.5) with the commutation relations of the deformed space, that is, calculating explicitly (5.4) using (5.5), leads to conditions on the coefficients $A_\mu^{\nu\rho_1\ldots\rho_j}$. In general a solution for those conditions is not unique.

In the previous chapter we saw that in the case of the θ-deformed space the commutation relation

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu \tag{5.6}$$

is compatible with the commutation relations $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$.

For the κ-deformed space there exist several sets of differential calculi which are all equivalent. Higher dimensional differential calculus is introduced in [15] and some of its application to field theory is discussed in [16, 17]. However, we do not wish to follow that approach here. Instead we continue with the analysis of (5.5). Requiring that the right-hand side of (5.5) is at most linear in the derivatives gives as one possible solution [18–20]

$$\begin{aligned} [\hat{\partial}_n, \hat{x}^\mu] &= \delta_n^\mu, \\ [\hat{\partial}_j, \hat{x}^\mu] &= \delta_j^\mu - ia\eta^{\mu n}\hat{\partial}_j, \end{aligned} \tag{5.7}$$

with $\eta^{\mu\nu} = diag(+,-,\ldots,-)$.

In the same way one can introduce Lorentz generators $M^{\mu\nu}$ acting on the κ-deformed space. Requiring the consistency with (5.3), that is,

$$\begin{aligned} M^{\rho\sigma}\left([\hat{x}^\mu, \hat{x}^\nu] - i\, C_\lambda^{\mu\nu}\hat{x}^\lambda\right) = \\ \left([\hat{x}^\mu, \hat{x}^\nu] - iC_\lambda^{\mu\nu}\hat{x}^\lambda\right) M^{\rho\sigma} = 0 \end{aligned} \tag{5.8}$$

gives

$$\begin{aligned} [M^{ij}, \hat{x}^\mu] &= \eta^{\mu j}\hat{x}^i - \eta^{\mu i}\hat{x}^j, \\ [M^{in}, \hat{x}^\mu] &= \eta^{\mu n}\hat{x}^i - \eta^{\mu i}\hat{x}^n + iaM^{i\mu}. \end{aligned} \tag{5.9}$$

We see that M^{ij} commute with coordinates as in the undeformed algebra,[1] while the generators M^{in} have deformed commutation relations with coordinates. We do not

[1] This is to be expected since M^{ij} are Lorentz generators in the undeformed directions.

refer to M^{in} as boost generators, since n is not the time direction, M^{in} include both boosts M^{0n} and rotations M^{bn}, $b = 1, \dots, n-1$.

Using the results obtained so far the κ-deformed Poincaré Hopf algebra (for a precise definition of Hopf algebras see Chaps. 7 and 8) can be written as

Algebra sector

$$
\begin{aligned}
[\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0, \\
[M^{ij}, \hat{\partial}_\mu] &= \delta^j_\mu \hat{\partial}^i - \delta^i_\mu \hat{\partial}^j, \qquad [M^{in}, \hat{\partial}_n] = \hat{\partial}^i, \\
[M^{in}, \hat{\partial}_j] &= \delta^i_j \frac{e^{2ia\hat{\partial}_n} - 1}{2ia} - \frac{ia}{2} \delta^i_j \hat{\partial}^l \hat{\partial}_l + ia \hat{\partial}^i \hat{\partial}_j, \\
[M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}.
\end{aligned} \tag{5.10}
$$

Coalgebra sector

$$
\begin{aligned}
\Delta\hat{\partial}_n &= \hat{\partial}_n \otimes 1 + 1 \otimes \hat{\partial}_n, \\
\Delta\hat{\partial}_j &= \hat{\partial}_j \otimes 1 + e^{ia\hat{\partial}_n} \otimes \hat{\partial}_j.
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
\Delta M^{ij} &= M^{ij} \otimes 1 + 1 \otimes M^{ij}, \\
\Delta M^{in} &= M^{in} \otimes 1 + e^{ia\hat{\partial}_n} \otimes M^{in} + ia\hat{\partial}_k \otimes M^{ik}.
\end{aligned} \tag{5.12}
$$

Counits and antipodes

$$
\varepsilon(\hat{\partial}_n) = \varepsilon(\hat{\partial}_j) = \varepsilon(M^{ij}) = \varepsilon(M^{in}) = 0, \tag{5.13}
$$

$$
\begin{aligned}
S(\hat{\partial}_n) &= -\hat{\partial}_n, \qquad S(\hat{\partial}_j) = -\hat{\partial}_j e^{-ia\hat{\partial}_n}, \\
S(M^{ij}) &= -M^{ij}, \\
S(M^{in}) &= -M^{in} e^{-ia\hat{\partial}_n} + iaM^{ik}\hat{\partial}_k e^{-ia\hat{\partial}_n} + ia(n-1)\hat{\partial}^i e^{-ia\hat{\partial}_n}.
\end{aligned} \tag{5.14}
$$

We see that the algebra sector (5.10) is deformed as well as the coalgebra sector (5.11), (5.12) leading to the deformed Leibniz rules for $\hat{\partial}_\mu$ and $M^{\mu\nu}$. This basis is called the bicrossproduct basis and was introduced for the first time in [12]. One can find a basis in which the algebra sector is undeformed, but the coalgebra sector remains deformed [18–20] and we present some details of construction in the following. In the classical limit, when the deformation parameter $a \to 0$ deformed Hopf algebras always reduce to the classical Poncaré Hopf algebra.

Let us look for a set of derivatives $\hat{D}_\mu$ which fulfill the consistency conditions (5.4) and have the following commutation relations with Lorentz generators

$$
[M^{\mu\nu}, \hat{D}_\mu] = \eta^\nu{}_\rho \hat{D}^\mu - \eta^\mu{}_\rho \hat{D}^\nu. \tag{5.15}
$$

Expressed in terms of the derivatives $\hat{\partial}_\mu$ (5.7) derivatives $\hat{D}_\mu$ read

$$
\hat{D}_n = \frac{1}{a}\sin(a\hat{\partial}_n) - \frac{ia}{2}\hat{\partial}^l \hat{\partial}_l e^{-ia\hat{\partial}_n}, \qquad \hat{D}_i = \hat{\partial}_i e^{-ia\hat{\partial}_n}; \tag{5.16}
$$

also

$$
\begin{aligned}
[\hat{D}_n, \hat{x}^n] &= \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu}, \\
[\hat{D}_n, \hat{x}^l] &= ia\hat{D}^l, \\
[\hat{D}_j, \hat{x}^n] &= 0, \\
[\hat{D}_j, \hat{x}^l] &= \delta_j^l \left(-ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu} \right).
\end{aligned} \tag{5.17}
$$

These derivatives are sometimes called Dirac derivatives in the literature and we keep that notation here. It is obvious that the Dirac derivatives are not linear derivatives, the right-hand side of (5.17) being a complicated function of $\hat{D}_\rho$. These complicated commutation relations lead to complicated Leibniz rules as well. Finally to summarize, the κ-deformed Poincaré Hopf algebra which we will use from now on is given by

Algebra sector

$$
\begin{aligned}
[\hat{D}_\rho, \hat{D}_\sigma] &= 0, \\
[M^{\mu\nu}, \hat{D}_\rho] &= \delta_\rho^\nu \hat{D}^\mu - \delta_\rho^\mu \hat{D}^\nu, \\
[M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}.
\end{aligned} \tag{5.18}
$$

Coproducts

$$
\begin{aligned}
\Delta M^{ij} &= M^{ij} \otimes 1 + 1 \otimes M^{ij}, \\
\Delta M^{in} &= M^{in} \otimes 1 + \frac{ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu}}{1 + a^2 \hat{D}_l \hat{D}^l} \otimes M^{in} \\
&\quad + \frac{ia\hat{D}_k}{1 + a^2 \hat{D}_l \hat{D}^l} \left(ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu} \right) \otimes M^{ik}, \\
\Delta \hat{D}_n &= \hat{D}_n \otimes \left(-ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu} \right) + \frac{ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu}}{1 + a^2 \hat{D}_l \hat{D}^l} \otimes \hat{D}_n \\
&\quad + ia \frac{\hat{D}_k}{1 + a^2 \hat{D}_l \hat{D}^l} \left(ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu} \right) \otimes \hat{D}^k, \\
\Delta \hat{D}_j &= \hat{D}_j \otimes \left(-ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu} \right) + 1 \otimes \hat{D}_j.
\end{aligned} \tag{5.19}
$$

Counits and antipodes

$$
\begin{aligned}
&\varepsilon(M^{ij}) = 0, \qquad S(M^{ij}) = -M^{ij}, \\
&\varepsilon(M^{in}) = 0, \qquad S(M^{in}) = -M^{in} e^{-ia\hat{\partial}_n} + iaM^{ik} \hat{\partial}_k e^{-ia\hat{\partial}_n} + ia(n-1)\hat{\partial}^i e^{-ia\hat{\partial}_n}, \\
&\varepsilon(\hat{D}_n) = 0, \qquad S(\hat{D}_n) = -\hat{D}_n + ia\hat{D}_k \hat{D}^k \frac{ia\hat{D}_n + \sqrt{1 + a^2 \hat{D}_\mu \hat{D}^\mu}}{1 + a^2 \hat{D}_l \hat{D}^l},
\end{aligned}
$$

$$\varepsilon(\hat{D}_j) = 0, \qquad S(\hat{\partial}_j) = -\hat{D}_j \frac{ia\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}}{1 + a^2\hat{D}_l\hat{D}^l}. \tag{5.20}$$

One sees from (5.18) that the algebra sector is undeformed (as it has been required), but the coalgebra sector (5.19) is deformed for both $M^{\mu\nu}$ and $\hat{D}_\rho$ generators. To be more precise, there is no deformation for the generators M^{ij}, because they are Lorentz generators for the undeformed dimensions. Since the commutation relations (5.18) between the generators $\hat{D}_\mu$ and $M^{\mu\nu}$ are undeformed, the representation content of the noncommutative field theory will be exactly the same as for its commutative correspondent. Note that that is not the case in the bicrossproduct basis since the commutation relations (5.10) between generators are deformed. For this reason the Dirac derivative (5.15)–(5.17) plays a special role in our approach.

5.3 Star product approach

So far we have worked in terms of the abstract algebra $\hat{\mathscr{A}}_{\hat{x}}$. To have a physical theory which makes predictions that can be checked by experiments we have to represent the abstract algebra on the space of commuting coordinates. This can be achieved by using the methods of deformation quantization, see Chap. 1. This method allows us to describe the properties of a noncommutative space in a perturbative way, order by order in the deformation parameter. In the zeroth order the commutative spacetime is obtained. In that way a new product is introduced. It is called a $\star$-product and it is a deformation of the usual pointwise product.

Using the Poincaré–Birkhoff–Witt property [21] which was discussed in Chaps. 1 and 4, in the case of κ-deformed space we obtain the following expression for the symmetrically ordered $\star$-product

$$\begin{aligned} f\star g(x) &= \lim_{\substack{z\to x\\ y\to x}} \exp\left(x^j\partial_{z^j}\left(\frac{\partial_n}{\partial_{z^n}} e^{-ia\partial_{y^n}}\frac{1-e^{-ia\partial_{z^n}}}{1-e^{-ia\partial_n}} - 1\right)\right.\\ &\qquad \left. + x^j\partial_{y^j}\left(\frac{\partial_n}{\partial_{y^n}}\frac{1-e^{-ia\partial_{y^n}}}{1-e^{-ia\partial_n}} - 1\right)\right) f(z)g(y) \\ &= f(x)g(x) + \frac{i}{2}C_\lambda^{\mu\nu}x^\lambda(\partial_\mu f)(\partial_\nu g) + \cdots . \end{aligned} \tag{5.21}$$

In zeroth order (5.21) is the same as the usual, commutative product and is a deformation of it. How this $\star$-product arises in the twist formalism is discussed in [22, 23].

An operator $\hat{O}$ acting on $\hat{\mathscr{A}}_{\hat{x}}$ can be represented by a differential operator $O^\star$ acting on $\mathscr{A}_x^\star$ which is now the algebra of commuting coordinates with the $\star$-product (5.21) instead of the usual pointwise multiplication:

$$
\begin{array}{ccc}
\hat{f}(\hat{x}) & \longmapsto & f(x) \\
\hat{o}\Big\downarrow & & \Big\downarrow O^\star \\
(\hat{O}\hat{f})(\hat{x}) & \longmapsto & (O^\star f)(x)\,.
\end{array}
\tag{5.22}
$$

In the previous chapter we used this method to find the $\star$-representation of the derivatives $\hat{\partial}_\mu$ acting on the θ-deformed space and we obtained

$$\partial_\mu^\star = \partial_\mu\,. \tag{5.23}$$

In the case of κ-deformed space the situation is more complicated. The $\star$-representation of the Dirac derivatives introduced in (5.16) and their Leibniz rules read:

$$
\begin{aligned}
D_n^\star f(x) &= \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{ia\partial_n^2}\partial_j\partial^j\right) f(x), \\
D_i^\star f(x) &= \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\partial_i f(x),
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
D_n^\star(f(x)\star g(x)) &= (D_n^\star f(x))\star(e^{-ia\partial_n}g(x)) \\
&\quad +(e^{ia\partial_n}f(x))\star(D_n^\star g(x)) \\
&\quad -ia\left(D_j^\star e^{ia\partial_n}f(x)\right)\star(D^{j\star}g(x)),
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
D_i^\star(f(x)\star g(x)) &= (D_i^\star f(x))\star(e^{-ia\partial_n}g(x)) \\
&\quad +f(x)\star(D_i^\star g(x)).
\end{aligned}
\tag{5.26}
$$

The $\star$-representation of the Lorentz generators $M^{\mu\nu}$ (5.9) can be obtained in this way as well [18–20].

5.4 Gauge theory on the κ-deformed space

In this section we introduce gauge theories on the κ-deformed space using the Seiberg–Witten approach [24, 25] which was discussed in the previous chapter.

Let us introduce the infinitesimal noncommutative gauge transformation as

$$\delta_\Lambda^{\mathrm{sw}}\psi = i\Lambda\star\psi(x), \tag{5.27}$$

where Λ is the noncommutative gauge parameter and ψ is the noncommutative matter field.

Following the Seiberg–Witten idea we suppose that the noncommutative gauge parameter can be expressed as a function of the commutative gauge parameter $\alpha = \alpha^a(x)T^a$ and the commutative gauge field $A_\mu = A_\mu^a T^a$, where T^a are generators of the Lie algebra of the gauge group. That is, we consider

$$\Lambda \equiv \Lambda_\alpha = \Lambda_\alpha(x;\alpha,A_\mu). \tag{5.28}$$

Then one uses the algebra relations[2]

$$\delta_\alpha^{\mathrm{sw}}\delta_\beta^{\mathrm{sw}} - \delta_\beta^{\mathrm{sw}}\delta_\alpha^{\mathrm{sw}} = \delta_{-i[\alpha,\beta]}^{\mathrm{sw}} \tag{5.29}$$

to calculate explicitly this functional dependence. Inserting $\Lambda_\alpha = \Lambda_\alpha(x;\alpha,A_\mu)$ in (5.29) gives[3]

$$(\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\star\psi + i(\delta_\alpha^{\mathrm{sw}}\Lambda_\beta - \delta_\beta^{\mathrm{sw}}\Lambda_\alpha)\star\psi = \delta_{-i[\alpha,\beta]}^{\mathrm{sw}}\psi. \tag{5.30}$$

The equation (5.30) can be solved perturbatively. For that one has to expand the $\star$-product. Since we are interested in the gauge theories on the κ-deformed space we use (5.21) and expand Λ_α as

$$\Lambda_\alpha = \alpha + a\Lambda_\alpha^1 + \cdots + a^k\Lambda_\alpha^k + \cdots. \tag{5.31}$$

Up to first order in the deformation parameter a the solution of (5.30) is

$$\Lambda_\alpha = \alpha - \frac{1}{4}x^\lambda C_\lambda^{\mu\nu}\{A_\mu,\partial_\nu\alpha\}. \tag{5.32}$$

Note that Λ_α is not Lie algebra valued. As in the case of the θ-deformed space the solution (5.32) is not unique, one can always add to it solutions of the homogeneous equation. Again, we do not discuss the nonuniqueness of the Seiberg–Witten map here. A more interested reader can find details in [26]. Using (5.27) and the solution for gauge parameter (5.32) one finds solution for the noncommutative matter field as well

$$\psi = \psi^0 - \frac{1}{2}x^\lambda C_\lambda^{\mu\nu}A_\mu(\partial_\nu\psi^0) + \frac{i}{8}x^\lambda C_\lambda^{\mu\nu}[A_\mu,A_\nu]\psi^0, \tag{5.33}$$

where ψ^0 is the commutative matter field, $\delta_\alpha\psi^0 = i\alpha\psi^0$.

If one compares $\star$-products for the θ-deformed space (Moyal $\star$-product) and for the κ-deformed space (5.21) one sees that up to first order in the deformation parameter they are of the same form (just replace $\theta^{\mu\nu}$ with $C_\lambda^{\mu\nu}x^\lambda$). Therefore it is not surprising that the solutions for Λ_α and ψ in the θ-deformed space can be obtained from (5.32) and (5.33) by replacing $C_\lambda^{\mu\nu}x^\lambda$ with $\theta^{\mu\nu}$ (and the other way around). However this analogy only applies in first order, in second order new terms will appear in the case of κ-deformed space compared to the θ-deformed space.

[2] Remember that because of (5.28) we can write $\delta_\alpha^{\mathrm{sw}} \equiv \delta_{\Lambda_\alpha}^{\mathrm{sw}}$ instead of $\delta_\Lambda^{\mathrm{sw}}$, see Section 4.4 for details.

[3] One should remember that now $\delta_\alpha^{\mathrm{sw}}\Lambda_\beta \neq 0$ because Λ_β depends on the commutative gauge field A_μ as well and $\delta_\alpha A_\mu = \partial_\mu\alpha - i[A_\mu,\alpha]$.

5.5 Gauge fields

In order to solve the Seiberg–Witten map for the gauge field V_μ one first has to choose $\partial^\star_\mu$ derivatives. In the θ-deformed space $\partial^\star_\mu = \partial_\mu$ is the most natural choice. In the κ-deformed space there are more possibilities. We choose $D^\star_\mu$ derivatives because of their vector-like transformation law (5.15). The covariant derivative[4] is introduced as $\mathscr{D}^\star_\mu \psi = D^\star_\mu \psi - iV_\mu \star \psi$. Under infinitesimal gauge transformations it transforms as follows

$$\delta^{\mathrm{sw}}_\alpha (\mathscr{D}^\star_\mu \psi) = i\Lambda_\alpha \star \mathscr{D}^\star_\mu \psi. \tag{5.34}$$

Then from (5.34) we obtain

$$\begin{aligned}(\delta^{\mathrm{sw}}_\alpha V_\mu)\star\psi &= D^\star_\mu(\Lambda_\alpha \star \psi) - \Lambda_\alpha \star (D^\star_\mu \psi) + i[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi \\ &\neq (D^\star_\mu \Lambda_\alpha)\star\psi + i[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi.\end{aligned}$$

The last line follows from the nontrivial Leibniz rules for $D^\star_\mu$ derivatives (5.25) and (5.26). In order to continue we split between n and j indices.

First we have a look at the j index:

$$\begin{aligned}(\delta^{\mathrm{sw}}_\alpha V_j)\star\psi &= D^\star_j(\Lambda_\alpha \star \psi) - \Lambda_\alpha \star (D^\star_j \psi) + i[\Lambda_\alpha \stackrel{\star}{,} V_j]\star\psi \\ &= (D^\star_j \Lambda_\alpha)\star e^{-ia\partial_n}\psi + i[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi,\end{aligned} \tag{5.35}$$

where we have used (5.26). In order to solve this equation we have to allow for V_j to be a differential operator instead of being a function as usual. We make the following ansatz

$$V_j \star \psi = \tilde{V}_j \star (e^{-ia\partial_n}\psi)$$

and insert it into (5.35). After using $e^{-ia\partial_n}(f\star g) = (e^{-ia\partial_n}f)\star(e^{-ia\partial_n}g)$ we find

$$\delta^{\mathrm{sw}}_\alpha \tilde{V}_j = (D^\star_j \Lambda_\alpha) + i\Lambda_\alpha \star \tilde{V}_j - i\tilde{V}_j \star \left(e^{-ia\partial_n}\Lambda_\alpha\right). \tag{5.36}$$

This equation can be solved order by order in the deformation parameter. The solution up to first order in a is

$$\begin{aligned}V_j = A_j - iaA_j\partial_n - \frac{ia}{2}\partial_n A_j - \frac{a}{4}\{A_n, A_j\} \\ + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \left(\{F_{\mu j}, A_\nu\} - \{A_\mu, \partial_\nu A_j\}\right),\end{aligned} \tag{5.37}$$

with the commutative field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$.

[4] Note that in this chapter we use a different notation for the covariant derivative, since the symbol $D^\star_\mu$ is reserved for the (non-covariant) Dirac derivative.

For V_n one follows the same steps, using the Leibniz rule for the $D^\star_n$ derivative (5.25) this time. The solution up to first order in a is

$$V_n = A_n - iaA^j\partial_j - \frac{ia}{2}(\partial_j A^j) - \frac{a}{2}A_jA^j + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \left(\{F_{\mu n}, A_\nu\} - \{A_\mu, \partial_\nu A_n\}\right). \tag{5.38}$$

From (5.37) and (5.38) we see that besides being enveloping algebra valued (a consequence of noncommutativity, that is, of the $\star$-product) the gauge field is also a differential operator, see the second term in (5.37) and (5.38). This is a consequence of the special properties of κ-deformed space, more concretely of the nontrivial Leibniz rules for $D^\star_\mu$ derivatives. Looking at (5.37) and (5.38) we see explicitly that the expansion in the enveloping algebra basis (4.58) and the expansion in the deformation parameter a (5.31) do not match. For example, the term $-\frac{ia}{2}(\partial_j A^j)$ in (5.38) is Lie algebra valued, but is a first-order term in a.

For completeness we repeat here also the solution for V_μ in the case of θ-deformed space

$$V_\rho = A_\rho + \frac{1}{4}\theta^{\mu\nu}\left(\{F_{\mu\rho}, A_\nu\} - \{A_\mu, \partial_\nu A_\rho\}\right). \tag{5.39}$$

This solution is not a differential operator since ∂_μ derivatives have the undeformed Leibniz rule.

Having the solutions of Seiberg–Witten map at hand, one calculates the field-strength tensor defined as

$$\mathscr{F}^\star_{\mu\nu} = i[\mathscr{D}^\star_\mu \stackrel{\star}{,} \mathscr{D}^\star_\nu]. \tag{5.40}$$

Since the gauge field V_μ is a differential operator it is not surprising that also the field strength tensor (5.40) is a differential operator. Therefore, we split the tensor $\mathscr{F}^\star_{\mu\nu}$ into "curvature-like" and "torsion-like" terms, like one usually does in gravity theories (but without any geometrical interpretation here)

$$\mathscr{F}^\star_{\mu\nu} = F^\star_{\mu\nu} + T^{\star\rho}_{\mu\nu}\mathscr{D}^\star_\rho + \cdots + T^{\star\rho_1\ldots\rho_l}_{\mu\nu} : \mathscr{D}^\star_{\rho_1}\ldots\mathscr{D}^\star_{\rho_l} : + \cdots, \tag{5.41}$$

where :: denote a basis in the algebra of covariant derivatives. For the action we will only use the "curvature-like" term $F^\star_{\mu\nu}$ and ignore all "torsion-like" terms. With this we have all the ingredients to write Lagrangian densities up to the first order in a, see [18–20] for details.

5.6 Integral and the action

To be able to write an action for noncommutative gauge theory we need an integral. First we try with the usual integral on the $n+1$ dimensional commutative space $\int \mathrm{d}^{n+1}x$. It should have the cyclic property

$$\int \mathrm{d}^{n+1}x\, f\star g = \int \mathrm{d}^{n+1}x\, g\star f = \int \mathrm{d}^{n+1}x\, f\cdot g. \tag{5.42}$$

This is required by the gauge invariance of the action for the gauge field. Namely, from $\delta_\alpha^{\mathrm{sw}} F^\star_{\mu\nu} = i[\Lambda_\alpha \stackrel{\star}{,} F^\star_{\mu\nu}]$ we have

$$\delta_\alpha^{\mathrm{sw}}(F^\star_{\mu\nu}\star F^{\star\,\mu\nu}) = i[\Lambda_\alpha \stackrel{\star}{,} F^\star_{\mu\nu}\star F^{\star\,\mu\nu}]. \tag{5.43}$$

Then, the action for the gauge field

$$S = \int \mathrm{d}^{n+1}x\, \mathrm{Tr}(F^\star_{\mu\nu}\star F^{\star\,\mu\nu}) \tag{5.44}$$

transforms as

$$\begin{aligned} \delta_\alpha^{\mathrm{sw}} \int \mathrm{d}^{n+1}x\, \mathrm{Tr}(F^\star_{\mu\nu}\star F^{\star\,\mu\nu}) & \qquad (5.45)\\ = \int \mathrm{d}^{n+1}x\, \mathrm{Tr}\left(i\Lambda_\alpha\star F^\star_{\mu\nu}\star F^{\star\,\mu\nu} - iF^\star_{\mu\nu}\star F^{\star\,\mu\nu}\star\Lambda_\alpha\right). \end{aligned}$$

Since the $\star$-product (5.21) is noncommutative (5.45) will be equal to zero (the action will be invariant) only if the integral is cyclic

$$\int \mathrm{d}^{n+1}x\, (f_1\star f_2\star\cdots\star f_k) = \int \mathrm{d}^{n+1}x\, (f_k\star f_1\star\cdots\star f_{k-1}). \tag{5.46}$$

Cyclic property is also important if we want to formulate the variational principle, see Chaps. 2 and 4.

In the canonically deformed space (5.42) is automatically fulfilled and the following analysis is not needed there. Unfortunately, in the case of κ-deformed space (5.42) is not fulfilled. One way to repair this is to introduce a measure function $\mu(x)$ such that

$$\int \mathrm{d}^{n+1}x\, \mu(x)\,(f\star g) = \int \mathrm{d}^{n+1}x\, \mu(x)\,(g\star f). \tag{5.47}$$

Expanding (5.47) up to first order in a one finds a condition on the measure function

$$\partial_\rho\left(C^{\rho\sigma}_\lambda x^\lambda \mu(x)\right) = 0, \tag{5.48}$$

that is

$$\partial_n\mu(x) = 0, \qquad x^j\partial_j\mu(x) = -n\mu(x). \tag{5.49}$$

It has been shown in [27] that for a given $\star$-product there always exists an equivalent $\star$-product[5] for which the integral (5.47) is cyclic to all orders. In our example of the symmetrically ordered $\star$-product (5.21) condition (5.48) ensures cyclicity of the

[5] Two $\star$-products $\star$ and $\star'$ are equivalent if there exists a map D such that $D = 1 + \mathcal{O}(h)$, with the deformation parameter h and

$$D(f\star g) = D(f)\star' D(g). \tag{5.50}$$

integral (5.47) to all orders in a [28, 29]. Equations (5.49) can be solved; however, the solution is not unique. Additionally, the solution for $\mu(x)$ is a independent so it does not vanish in the commutative limit $a \to 0$. This means that it will spoil the classical limit of the theory (equations of motion for example). And because of its explicit x-dependence[6] it will break the κ-Poincaré invariance of the integral.

Besides the one just described, there are other notions of integration, each of them with some interesting properties. For example, one can relax the cyclicity condition and construct an integral which is κ-Poincaré covariant by using the quantum trace method [28]. The integral obtained in that way does not have the cyclic property, therefore, it is not convenient for analyzing gauge theories. An integral that is quasi-cyclic is defined in [30], but its application to field theory is still to be analyzed.

So far there has not been a completely satisfactory answer to the question of proper definition of the integral on κ-deformed space. In the case of $U(1)$ gauge theory a gauge-invariant action is constructed in [31], but the analysis is still far from being complete. For some work concerning formulations of quantum field theory on the κ-Minkowski space see [32, 33] and references therein. Attempts to apply Noether theorem and construct conserved charges in κ-Minkowski space are made in [16, 17] and in [34, 35]. However, the construction of conserved quantities when a deformed symmetry is present is still an open question and a subject of ongoing research, see for example [36, 37].

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[6] One possible solution for $\mu(x)$ is $\mu = \frac{1}{x^0 x^1 \ldots x^{n-1}}$.

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Chapter 6
Noncommutative Spaces

Fedele Lizzi

In this chapter we present some of the basic concepts needed to describe noncommutative spaces and their topological and geometrical features. We therefore complement the previous chapters where noncommutative spaces have been described by the commutation relations of their coordinates. The full algebraic description of ordinary (commutative) spaces requires the completion of the algebra of coordinates into a $C^\star$-algebra, this encodes the Hausdorff topology of the space. The smooth manifold structure is next encoded in a subalgebra (of "smooth" functions). Relaxing the requirement of commutativity of the algebra opens the way to the definition of noncommutative spaces, which in some cases can be a deformation of an ordinary space. A powerful method to study these noncommutative algebras is to represent them as operators on a Hilbert space. We discuss the noncommutative space generated by two noncommuting variables with a constant commutator. This is the space of the noncommutative field theories described in this book, as well as the elementary phase space of quantum mechanics. The Weyl map from operators to functions is introduced in order to produce a $\star$-product description of this noncommutative space.

6.1 Commutative geometry (and topology)

In Hilbert's foundations of geometry [1] the concepts of points, lines, and planes are considered intuitive and no attempt is made to define them. These "undefined" points are nevertheless the basis of any topological space, differentiable manifold, bundle, and so on, all geometrical concepts built on spaces made of points. This gave the impression that geometrical notions cannot survive without points. Quantum mechanics forced a change of this attitude. While in classical mechanics the state of a system can be described by a point in a phase space, Heisenberg's uncertainty principle makes the concept impossible in quantum mechanics. This led von Neumann [2] to speak of *pointless* geometry.

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In the following we introduce the basic mathematics of noncommutative geometry at an unspecialized level, that of a high-energy physics student for example. Sometimes we sacrifice rigor and refer to some of the classic reference books [3–6] for details and proofs. An extended and more rigorous treatment of the topics of this chapter will also appear in the forthcoming book [7].

In the present section we discuss ordinary topology and geometry from a point of view which enables its generalization to a noncommutative setting. The main tool is the transcription of the usual geometrical concepts in terms of algebras of operators. The starting point is a series of theorems due to Gel'fand and Naimark (for a review see for example [4, 5]). They established a complete equivalence between *Hausdorff topological spaces* and *commutative C^*-algebras*. From a physicist point of view one can look at this activity as describing the topology (and geometrical properties) of a space not seeing it as a set of points, but as the set of fields defined on it. In this sense the tools of noncommutative geometry resemble the methods of modern theoretical physics.

6.1.1 Topology and algebras

A topological space M is a set on which a *topology* is defined: a collection of open subsets obeying certain conditions, this enables the concept of convergence of succession of points $x_n \in M$ to a limit point $x = \lim_n x_n$. Together with the concept of convergence goes the notion of continuous function. A function from a topological space into another topological space is continuous if the inverse image of an open set is open, but as a consequence it maps convergent sequences into convergent sequences:

$$\lim_n f(x_n) = f(x). \tag{6.1}$$

A Hausdorff topology makes the space separable, i.e., given two points it is always possible to find two disjoint open sets each containing one of the two points. The common topological spaces encountered in physics (for example, manifolds) are separable.

Of particular interest in this context is the set of *complex-valued* continuous function. They form a commutative algebra because the sum or product of two continuous functions is still continuous. We will show how it is possible to define the topology of a space from the algebra of continuous functions on it. Moreover, we will show how to construct the topological space starting from the abstract algebra. On one hand every Hausdorff topological space defines naturally a commutative algebra, the algebra of continuous complex-valued functions over it. Remarkably, under certain technical assumptions spelled below, the reverse is also true, i.e., given a commutative algebra $\mathscr{A}$ as an abstract entity, it is always possible to find a topological space whose algebra of continuous functions is $\mathscr{A}$. Therefore, we can establish a complete equivalence between topological spaces and algebras. In the following we will describe these mathematical structures from an "user" point of view, keeping the technicalities at a minimum and refer the literature for proofs and details.

The technical assumptions we have mentioned are resumed in the fact that the algebra $\mathscr{A}$ must be a C^*-algebra. This is, first of all, a vector space with the structure of an associative algebra over the complex numbers $\mathbb{C}$, i.e., a set on which we can define two operations, sum (associative and commutative) and product (associative but not necessarily commutative), and the product of a vector by a complex number, with the following properties:

(1) $\mathscr{A}$ is a vector space over $\mathbb{C}$, i.e., $\alpha a+\beta b\in\mathscr{A}$ for $a,b\in\mathscr{A}$ and $\alpha,\beta\in\mathbb{C}$.
(2) It is distributive over addition with respect to left and right multiplication, i.e., $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$, $\forall\, a,b,c\in\mathscr{A}$.

$\mathscr{A}$ is further required to be a Banach algebra:

(3) It has a norm $\|\cdot\|:\mathscr{A}\to\mathbb{R}$ with the usual properties

a) $\|a\|\geq 0\,,\quad \|a\|=0 \iff a=0$
b) $\|\alpha a\|=|\alpha|\,\|a\|$
c) $\|a+b\|\leq\|a\|+\|b\|$
d) $\|ab\|\leq\|a\|\,\|b\|$

The Banach algebra $\mathscr{A}$ is called a $*$-algebra if, in addition to the properties above, it has a hermitian conjugation operation $*$ (analogous to the complex conjugation defined for $\mathbb{C}$) with the properties

(4) $(a^*)^*=a$
(5) $(ab)^*=b^*a^*$
(6) $(\alpha a+\beta b)^*=\bar{\alpha}a^*+\bar{\beta}b^*$
(7) $\|a^*\|=\|a\|$
(8) $\|a^*a\|=\|a\|^2$

for any $a,b\in\mathscr{A}$ and $\alpha,\beta\in\mathbb{C}$, where $\bar{\alpha}$ denotes the usual complex conjugate of $\alpha\in\mathbb{C}$. Finally,

(9) It is complete with respect to the norm.

C^*-algebras play a very important role in mathematics because as we will see their study is basically the study of topology. A good introduction to their properties is found in the book [8].

Example 6.1.
Examples of C^*-algebras are $n\times n$ matrices, bounded operators on an infinite-dimensional Hilbert space, as well as compact operators. The norm is the supremum norm in all these cases. These are noncommutative, examples of commutative algebras are $\mathbb{C}$ itself, or the continuous functions on the plane. Note that several commonly used algebras do not satisfy all of the definitions. For example, the set of upper triangular matrices does not have the hermitian conjugation, trace class operators are not complete, and the Hilbert space of L^2 functions has a norm which does not satisfy item (8) above. □

Every Hausdorff topological space has a natural *commutative* C^*-algebra associated with it: the algebra of continuous complex-valued functions. If the space is compact this algebra contains the unity $\mathbb{I}$ and is called unital. The converse is also true. Every unital commutative C^*-algebra is the C^*-algebra of continuous functions on some compact topological space. Nonunital algebras are similarly associated with noncompact Hausdorff spaces.

6.1.2 Reconstructing the space from the algebra

We now show how the topological space can be reconstructed from the algebra. We first introduce the notion of *state*. A state is a linear functional from a C^*-algebra $\mathscr{A}$ (not necessarily commutative) into complex numbers:

$$\phi : \mathscr{A} \longrightarrow \mathbb{C}, \tag{6.2}$$

with the positivity and normalization requirements

$$\phi(a^*a) \geq 0 \ \ \forall a \in \mathscr{A}, \ \ \|\phi\| = 1 \ . \tag{6.3}$$

In this case the norm is defined as

$$\|\phi\| = \sup_{\|a\| \leq 1} \{\phi(a)\} \ . \tag{6.4}$$

If the algebra is unital $\phi(\mathbb{I}) = 1$.

The space of states is convex, i.e., any linear combination of states of the kind $\cos^2\lambda\phi_1 + \sin^2\lambda\phi_2$ is still a state for any value of λ. Some states cannot be expressed as such convex sum, they form the boundary of the set and are called *pure states*.

Example 6.2.
Consider the case of $n \times n$ complex-valued matrices. A state is given by a matrix (which with an abuse we still call ϕ) with the definition

$$\phi(a) = \mathrm{Tr}\phi a. \tag{6.5}$$

Positivity requires the matrix ϕ to be self-adjoint with positive eigenvalues, and normalization requires it to have unit trace. Since the matrix is self-adjoint it can be diagonalized. There are two possibilities. Either more than one eigenvalues is different from zero, and in this case it is immediate to see that we can write it as the convex sum of two diagonal matrices of trace 1. Alternatively only one eigenvalue is different from zero, and it must be the unity. In this case it is not possible to express ϕ as the convex sum of two matrices of trace 1, since positivity requires diagonal elements to be positive numbers less than 1. So pure states are nothing else but pure density matrices, which correspond to the projectors, these in turn are in a *one-to-one correspondence with the normalized n-dimensional vectors (the rays). This construction can be carried over to the infinite-dimensional case, considering bounded operators on an infinite-dimensional separable Hilbert space.* □

Consider next a commutative algebra and its set of pure states. We can give to this set a *topology* as follows: given a succession of pure states δ_{x_n} we find its limit as

$$\lim_n \delta_{x_n} = \delta_x \Leftrightarrow \lim_n \delta_{x_n}(a) = \delta_x(a) \ \ \forall a \in \mathscr{A} \ . \tag{6.6}$$

We have constructed the topological space associated with the C^*-algebra $\mathscr{A}$. We have therefore a *duality* between topological spaces and C^*-algebras: a topological space determines the C^*-algebra of its continuous complex-valued functions. Conversely any commutative C^*-algebra, using uniquely algebraic techniques, determines a topological space whose algebra of continuous functions is the initial C^*-algebra.

The reconstruction of the topological space from the algebra via the set of pure states is one of various equivalent ways to obtain the space from the algebra. It is worth to briefly comment on some of the alternatives since in the noncommutative case these are not anymore the same and capture different aspects of the noncommutative geometry. For commutative algebras it turns out that the space of pure states is the same as the state of irreducible one-dimensional representations. It is possible to give a topology (called regional topology) [9] directly on the space of representations of an algebra, and in the commutative case this topology is the same as the one described earlier. In this case the space of points is also the same as the space of *maximal ideals* of the algebra. An ideal of an algebra is a subalgebra $\mathscr{I}$ with the property that

$$ab \in \mathscr{I} \ \forall a \in \mathscr{A}, \ \forall b \in \mathscr{I} \ , \tag{6.7}$$

the relevant example of ideal for the algebra of functions on some space is the set of functions vanishing in some closed set. Recall that if a continuous function vanishes on some set of a topological space, it will vanish also on the closure of the set, therefore the structure of ideals *feels* the topology of the underlying space. A maximal ideal is an ideal which is not contained in any other ideal (and is not the whole algebra). Since the ideal of functions vanishing in a given set is contained in the ideal of functions vanishing in any smaller set contained in the first set, it is intuitively obvious that the functions vanishing at a given point are an ideal not contained in any other ideal, hence the one-to-one correspondence between points and maximal ideals. A topology based on the closure of the set of ideals can be given (called hull-kernel topology), thus giving a third (equivalent) manner to reconstruct a space from a C^*-algebra. We have seen three different sets that we can build exclusively form the algebra:

- pure states
- irreducible (one-dimensional) representations
- maximal ideals

On this set we can build, purely algebraically, three topologies, which turn out to be same for commutative algebras.

6.1.3 Geometrical structures

We have mainly dealt so far with the topology of Hausdorff spaces, which we can call, in the spirit of these notes, *commutative spaces*. What about the other geometrical structures? We can transcribe all standard concepts of geometry at the algebraic level, as properties of C^*-algebras and of other operators. This program, started by Connes [3], has been going for some time in the construction of some sort of *dictionary* transcribing the concepts of commutative geometry into concepts connected to C^*-algebras. The aim of this exercise should be evident: once we have translated pointwise geometry into operations at the algebraic level, these are more robust and can still be used at the level of noncommutative C^*-algebras, thus describing a noncommutative geometry. Let us give a few entries of this continuously evolving dictionary.

The presence of a smooth structure, i.e., a manifold structure, is equivalent to considering a subalgebra $\mathscr{A}_\infty \subset \mathscr{A}$ of "smooth" functions. This subalgebra can be given the structure of a Fréchet algebra, which is a locally convex algebra with its topology generated by a sequence of seminorms $\|\cdot\|_k$ which separate points: that is, $\|a\|_k = 0 \forall k \Leftrightarrow a = 0$. The seminorms for this algebra are

$$\|a\|_k = \sup_{x\in M}\{ \, |\partial^\alpha a(x)| \text{ for } |\alpha| \le k \, \} \; . \tag{6.8}$$

A theorem of Serre and Swan establishes an equivalence between bundles and modules. A bundle E over topological space M (called the base) is a triple composed by E (which is also a topological space), M, and a continuous surjective map $\pi : E \to M$ and such that for each $x \in M$ the space $\pi^{-1}(x)$ is homeomorphic to a space F, called the typical fiber. When F is a vector space we have a *vector bundle*. Locally the bundle is trivial, i.e., there is a covering U_i of M such that locally $\pi^{-1}(U_i) = U_i \times F$. A *section* of a bundle is a map $s : E \to M$ such that $\pi \circ s = \mathrm{id}_\mathrm{M}$. Examples of bundles abound in physics, often with the further structures, like fiber bundles, which are vector bundles together with the action of a group G on the fiber F. Yang–Mills fields are sections of fiber bundles. It turns out that a vector bundle over a manifold M is completely characterized by its space of smooth sections $\mathscr{E} = \Gamma(E,M)$.

It is possible to substitute the concept of bundle with the one of *projective module*. A left module $\mathscr{E}$ is a vector space over $\mathbb{C}$ on which the algebra acts, that is, for $a,b \in \mathscr{A}$, $\eta, xi \in \mathscr{E}$ we have

$$a\eta \in \mathscr{E} \tag{6.9}$$

and

$$(ab)\eta = a(b\eta) \; , \;\; (a+b)\eta = a\eta + b\eta \; , \;\; a(\eta+\xi) = a\eta + a\xi \; . \tag{6.10}$$

The definition of right module is analogous. We have purposive used the same symbol for the sections of a bundle and for the module, since the latter is a relevant example of the former, where the algebra is the algebra of continuous functions over the base. A module $\mathscr{E}$ is finite if it is generated by a finite number of generators, and it is projective if given any two other modules $\mathscr{M}$ and $\mathscr{N}$, and a homomorphism $\phi : \mathscr{M} \to \mathscr{N}$ connecting them, then any surjective homomorphism $\phi_M : \mathscr{E} \to \mathscr{M}$ can be lifted to a homomorphism such that $\phi_N = \phi \circ \phi_M$. What this means actually is quite simple, it is saying that, heuristically, any finite projective module is made of matrices with elements in the algebra. The Serre–Swan theorem then says that any finite projective module over the algebra of smooth functions is isomorphic to the space of sections of a bundle and that conversely the space of sections of a bundle is isomorphic to a finite projective bundle. This means that it is always possible to write the space of sections $\Gamma(E,M) = p\mathscr{A}^N$ where p is a matrix of elements of the algebra $p \in M_N(\mathscr{A})$ with the property that $p^2 = p$.

The transcription in algebraic terms of geometry comprises several more entries. Differential forms are realized as operators with the help of a generalized Dirac operator D, integrals of functions are calculated as traces of the corresponding operators, and the list goes on to comprise several more entries. It is possible to characterize a manifold, given by an algebra $\mathscr{A}_\infty$ with its differential structure, given by the generalized Dirac operator D, exclusively in algebraic terms [10]. The dimensionality is encoded in the growth of the eigenvalues of D, differentiability is given by multiple commutators of the elements of the algebra with D, as well as the domain of D^m acting on the Hilbert space on which $\mathscr{A}_\infty$ is represented. There are other conditions which mirror smoothness. We refer to the cited literature for details of this and the other entries of the dictionary and proceed to the generalization of this commutative geometry to noncommutative spaces.

6.2 Noncommutative spaces

In the previous section we have established the one-to-one correspondence between commutative C^*-algebras and ordinary Hausdorff spaces and we have shown how to reconstruct the points using purely algebraic methods. It now is possible to go beyond commutativity and define a noncommutative space as the object described by a *noncommutative* C^*-algebra. One can now ask if there are still points and a topology to recognize in this novel setting. In general we can still recognize a set of pure states, of representations (possibly of dimension larger than one), and of maximal ideals (now one has to distinguish among left, right, or bilateral ideals). These spaces now do not coincide anymore. Moreover, the algebra of continuous functions on the "points", being commutative, cannot anymore be the starting algebra. The concept of point becomes evanescent, and in some cases one is forced to abandon it altogether. Take for example the set of $n \times n$ complex matrices. It has only one representation (n-dimensional), but not one-dimensional representations. It has n unitarily equivalent pure states and no maximal ideals (apart from the whole algebra). One could be tempted to say that it describes a single point, but there is more structure in this algebra than in its commutative counterpart (complex numbers). The same can be said in the infinite-dimensional case of compact operators. We will see below that the equivalence of these algebras with $\mathbb{C}$ as far as the representations are concerned is captured by Morita equivalence.

6.2.1 The GNS construction

Still it is possible to do geometry, noncommutative geometry. This means that we extract geometric information directly form the algebra. The main technique is to represent the C^*-algebra in a Hilbert space. There is another result due to Gel'fand and Naimark (and Segal) which states that any C^*-algebra can be represented as bounded operators on a Hilbert space, and this of course strikes a chord in the hearth of physicists! The proof is constructive, namely, given a C^*-algebra one has a natural procedure (called GNS construction) to build a Hilbert space on which the algebra acts as bounded operators, with the C^* norm given by the operatorial norm.

The GNS construction is based on the fact that since every algebra has an obvious action on itself, we can consider the algebra itself as the starting vector space for the construction of the Hilbert space. To make this space a Hilbert space we first need an inner product with certain properties, and then we need to complete in the norm given by this product. Note that the Hilbert space norm is not the original norm of the C^*-algebra.

First we note that any state ϕ gives a bilinear map with some of the properties of inner product: $\phi(a^*b)$. The problem with this map is that there may be instances in which $\phi(a^*a)$ is zero, even if a is not the null vector. To this end consider the space of null elements defined as,

$$\mathscr{N}_\phi = \{a \in \mathscr{A} \mid \phi(a^*a) = 0\} \ . \tag{6.11}$$

This space turns out to be a left ideal. This can be proven using the relation

$$\phi(a^*b^*ba) \leq \|b\|^2 \phi(a^*a) \ , \tag{6.12}$$

so that $a \in \mathscr{N}_\phi \Rightarrow ba \in \mathscr{N}_\phi \, \forall b \in \mathscr{A}$. This ideal of null states can be eliminated by considering the space of equivalence classes of the elements of $\mathscr{A}$ up to elements of $\mathscr{N}_\phi$. We can then equip this space with the scalar product

$$\langle [a],[b] \rangle_\phi = \phi(a^*b) \ . \tag{6.13}$$

This product is by definition independent from the representative of the equivalence class. It defines a norm, and the Hilbert space is the topological completion of the space of equivalence classes with respect to this norm.

The algebra $\mathscr{A}$ is naturally represented on the Hilbert space by associating to any element $a \in \mathscr{A}$ an operator $\hat{a}$ with action

$$\hat{a}[b] = [ab] \ , \tag{6.14}$$

and again the action does not depend on the representative.

Thus, we have a representation of our algebra on the Hilbert space. The operators corresponding to the elements of $\mathscr{A}$ are bounded, in fact, expressing with

$$\|\hat{a}\|_\phi = \sup_{\phi(b^*b)\leq 1} \phi(b^* a^* ab), \tag{6.15}$$

we have the operator norm on the Hilbert space, using (6.12)

$$\|\hat{a}[b]\|^2 = \phi(b^* a^* ab) \leq \|a\|^2 \phi(b^* b), \tag{6.16}$$

and considering the supremum over $\phi(b^*b) \leq 1$ one obtains $\|\hat{a}\|_\phi \leq \|a\|$. Therefore, since all operators of a C^*-algebra have finite norms, $\hat{a}$ is a bounded operator on the Hilbert space $\mathscr{H}_\phi$ that we have just built. Note that the association of an operator to the element of the algebra depends on the choice of the state ϕ.

Conversely, given an algebra of bounded operators on a Hilbert space, any normalized vector $|\xi\rangle$ defines a state with the expectation value

$$\phi_\xi(a) = \langle \xi | \hat{a} | \xi \rangle \ . \tag{6.17}$$

It results that to any state ϕ it corresponds a *vector state*, i.e., there is a vector $\xi_\phi \in \mathscr{H}_\phi$ such that

$$\left\langle \xi_\phi \middle| \hat{a} \middle| \xi_\phi \right\rangle = \phi(a) \,. \tag{6.18}$$

The vector ξ_ϕ is defined by

$$\xi_\phi := [\mathbb{I}] = \mathbb{I} + \mathscr{N}_\phi \tag{6.19}$$

and is readily seen to verify (6.18). Furthermore, the set $\{\pi_\phi(a)\xi_\phi \mid a \in \mathscr{A}\}$ is just the dense set $\mathscr{A}/\mathscr{N}_\phi$ of equivalence classes. This fact is encoded in the definition of *cyclic vector*. The vector ξ_ϕ is cyclic for the representation $(\mathscr{H}_\phi, \pi_\phi)$. By construction, a cyclic vector is of norm one: $\|\xi_\phi\|^2 = \|\phi\| = 1$.

The cyclic representation $(\mathscr{H}_\phi, \pi_\phi, \xi_\phi)$ is unique up to unitary equivalence. It can be shown that this representation of the algebra is irreducible if ϕ is a pure [11] state.

Example 6.3.
Let us consider the example of the commutative algebra of continuous functions on the real line vanishing at infinity. Choosing as pure state

$$\delta_{x_0}(a) = a(x_0), \tag{6.20}$$

the null space is given by all functions vanishing at x_0. The inner product is then given by

$$\langle a, b \rangle_\delta = a(x_0)^* b(x_0) \ , \tag{6.21}$$

and the Hilbert space turns out to be just $\mathbb{C}$. The algebra acts on this space by multiplication of complex numbers:

$$\hat{a}[b] = a(x_0) b(x_0) \ . \tag{6.22}$$

We should not be surprised of the fact that the Hilbert space is $\mathbb{C}$, the state is pure, and the only irreducible representations of a commutative algebra are one-dimensional.

The situation is different if we choose a non-pure state, for example,

$$\phi(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \mathrm{e}^{-x^2} a(x). \tag{6.23}$$

This time there are no nonzero elements of the algebra such that $\phi(a^* a) = 0$. The Hilbert space therefore contains the continuous functions, the inner product is given by

$$\langle a, b \rangle_\phi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \mathrm{e}^{-x^2} a^*(x) b(x). \tag{6.24}$$

The completion of this space gives the space $L^2(\mathbb{R})$ with a gaussian measure. Then the operator representation of the algebra is just given by the pointwise multiplication of functions

$$\hat{a} b(x) = a(x) b(x) \; . \tag{6.25}$$

□

Example 6.4.
Let us give a noncommutative example: the matrix algebra $\mathbb{M}_2(\mathbb{C})$ with the two pure states

$$\phi_1 \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{11} \; , \quad \phi_2 \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{22} \; . \tag{6.26}$$

The corresponding representations are equivalent, being indeed both equivalent to the defining two-dimensional one. The ideals of elements of "vanishing norm" of the states ϕ_1, ϕ_2 are, respectively,

$$\mathcal{N}_1 = \left\{ \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \right\} , \quad \mathcal{N}_2 = \left\{ \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \right\} . \tag{6.27}$$

The associated Hilbert spaces are then found to be

$$\begin{aligned} \mathcal{H}_1 &= \left\{ \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} \right\} \simeq \mathbb{C}^2 = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} , \; \langle X, X' \rangle = x_1^* x_1' + x_2^* x_2' \; , \\ \mathcal{H}_2 &= \left\{ \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \end{bmatrix} \right\} \simeq \mathbb{C}^2 = \left\{ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\} , \; \langle Y, Y' \rangle = y_1^* y_1' + y_2^* y_2' \; . \end{aligned} \tag{6.28}$$

As for the action of any element $A \in \mathbf{M}_2(\mathbb{C})$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, we have

$$\pi_1(A) \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 & 0 \\ a_{21} x_1 + a_{22} x_2 & 0 \end{bmatrix} \equiv A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

$$\pi_2(A)\begin{bmatrix}0 & y_1\\ 0 & y_2\end{bmatrix} = \begin{bmatrix}0 & a_{11}y_1 + a_{12}y_2\\ 0 & a_{21}y_1 + a_{22}y_2\end{bmatrix} \equiv A\begin{pmatrix}y_1\\ y_2\end{pmatrix}. \tag{6.29}$$

The two cyclic vectors are given by

$$\xi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_2 = \begin{pmatrix}0\\1\end{pmatrix}. \tag{6.30}$$

The equivalence of the two representations is provided by the off-diagonal matrix

$$U = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \tag{6.31}$$

which interchanges 1 *and* 2 *:* $U\xi_1 = \xi_2$*. Since* π_1 *and* π_2 *are irreducible representations and since any nonvanishing vector is a cyclic vector if the representation is irreducible, we see that* π_1 *and* π_2 *are unitary equivalents and can therefore be identified.* □

6.2.2 Commutative and noncommutative spaces

Sometimes, even in the presence of a noncommutative algebra, we are still in the presence of an ordinary space. Consider functions from a manifold into $n \times n$ complex-valued matrices. In this case the algebra can obviously still be associated with the original manifold, and we cannot really talk of a noncommutative geometry. Note that in this case, since algebra of $n \times n$ matrices has only one representation, we have one representation for each point of the original manifold, as in the commutative case. There are more pure states, as in (6.26), but they are unitarily equivalent. It is like we had points, but with an inner structure. This is sometimes refereed to as an “almost commutative geometry”.

This characteristic is captured by the concept of (strong) Morita equivalence [12]. Two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent if there exists a complex vector space $\mathscr{E}$ which is a left module for $\mathscr{A}$ and a right one for $\mathscr{B}$. In $\mathscr{E}$ two inner products,[1] are defined with values in the two algebras, such that the representations are continuous and bounded, and with the property

$$\langle \eta, \xi \rangle_{\mathscr{A}}\, \chi = \eta\, \langle \xi, \chi \rangle_{\mathscr{B}} \quad \forall\, \eta, \xi, \chi \in \mathscr{E}. \tag{6.32}$$

The important property of Morita equivalent algebras is that they have the same space of (classes of unitary inequivalent) representations with the same topology. In particular all algebras, Morita equivalent to commutative algebras, are algebras of function from some Hausdorff topological space which can be uniquely reconstructed. Morita equivalent algebras also have the same (algebraic)

[1] There are other requirements of continuity and density for the definition. The two inner products are sesquilinear forms with the usual properties. For details see [5].

K-theory. Hence in some sense two Morita equivalent algebras are algebras of functions on the same "space".

6.2.3 Deformations of spaces

There are several noncommutative spaces that have been studied: deformations (with one or more continuous parameters) of commutative algebras of functions on a topological space, or algebras of matrix-valued functions on commutative spaces. There are then truly noncommutative structures that are not linked to a "classical" (commutative) manifold. In some instances these NC algebras can be associated with non-Hausdorff spaces, the typical example being that of torus foliations for irrational theta. Similarly examples of spaces with a nonseparating topology in which a finite set of points can keep track of the homotopy of the original space are described in [13].

The standard example of a genuine noncommutative geometry is the *noncommutative torus* [3, 5] which we now briefly describe. In Fourier transform one can write functions on the torus (characterized by $x_i \in [0,1]$) as

$$f(x) = \sum f_{mn} U_1^n U_2^m, \tag{6.33}$$

with $U_i = e^{2\pi i x_i}$ and obviously $U_1 U_2 = U_2 U_1$. In this setting continuous functions are the ones with coefficients such that $\lim_{n_i \to \pm\infty} f_{n_1 n_2} \to 0$ faster than n_i^{-2}. From this C^*-algebra it is possible to reconstruct the torus as a topological space as shown in the previous section. If one now generalizes the commutation relation of the U's to the case

$$U_1 U_2 = e^{i2\pi\theta} U_2 U_1, \tag{6.34}$$

the algebra generated by (6.33) is a noncommutative algebra; it describes a deformation of the torus called noncommutative torus. When θ is irrational there is no ordinary space underlying it, in this case we are in the presence of a truly noncommutative space. The name noncommutative torus is given to various completions, with different norms, of the algebra (6.33) with the relation (6.34), corresponding to functions continuous, differentiable, analytic, etc. They all correspond to the various classes of functions of a "manifold" whose coordinates obey the commutation relation $[x_1, x_2] = i\theta$. It should however be kept in mind that this is just an heuristic view, as it is impossible to talk of a topological space in this case. We do not have the points of the space in this case!

Noncommutative tori are very different mathematical structures in the cases of θ rational or irrational. In the first case, $\theta = p/q$, p, q integers, the noncommutative torus is Morita equivalent to the algebra of functions on the ordinary torus [14], they are in fact isomorphic to the algebra of $q \times q$ matrices on a torus. In the irrational case the algebra does not describe any Hausdorff topological space. It can be seen that they describe the space of orbits of the points of a circle under the action of rotation of an angle $2\pi\theta$. As is known every orbit is dense, and therefore in the neighborhood of any point there is the whole space. If one considers then the circle quotiented by these rotations the Hausdorff topology would give a single point. Likewise if we consider functions which are constant on any given orbit, we obtain only functions constant on the circle. In noncommutative geometry there is a well-defined procedure, called the crossed product, which, starting from the action of functions on a manifold and the action of a group on it, gives the algebra on the quotient space. The application of this procedure to the case of the circle with the action of discrete irrational rotations gives the algebra of the noncommutative torus. Hence the noncommutative algebra captures more structure. In general noncommutative tori with parameters θ and θ' are Morita equivalent iff the parameters are connected by a $SL(2,Z)$ transformation:

$$\theta' = \frac{a\theta+b}{c\theta+d} \ , \ ad-bc=1, \tag{6.35}$$

with a,b,c,d integers.

Finally, a relevant example of noncommutative spaces is that of quantum groups and Hopf algebras, discussed in the next chapter.

6.3 The noncommutative geometry of canonical commutation relations

The original example of a noncommutative space is quantum phase space; this is a well-established concept from the early days of quantum mechanics, with $\hbar$ a dimensionful quantity, with the dimensions of the area of the phase space of a one-dimensional particle. It is a "small" parameter, in the sense that in the limit in which it goes to zero, classical mechanics should emerge. In the usual view, for example in the courses of the standard physics curriculum, quantum and classical mechanics, however, are two different theories, using different mathematical tools, and the passage from one to the other (the classical limit) is not an immediate and unambiguous procedure. In reality there is a procedure, *deformation quantization*, which connects the two. In this case quantum mechanics is seen as a deformation of the classical theory, and the two theories are both seen as a theory of states on the $*$-algebra of observables. The crucial difference between the two theories is that in the quantum case the algebra is noncommutative.

The geometry underlying Hamiltonian classical mechanics is a Poisson (symplectic) geometry. The space of position and momenta, the *phase space*, is equipped with a Poisson bracket, and time evolution is generated by a Hamiltonian vector field. The set of functions on phase space is the set of observables of the theory: position, momentum, angular momentum, energy, temperature, etc. Under the conditions described in the first sections of this chapter it is possible to reconstruct the phase space from these observables. It is important that we can shift the emphasis from the points of phase space to the observables. The points are then the (pure) states of the algebra of observables.

Quantum mechanics forces the loss of the classical phase space; positions and momenta are substituted by *noncommuting* self-adjoint operators. We like to say that quantization is the rendering of a classical phase space a noncommutative geometry. In this section we will discuss the quantum phase space of a one-dimensional particle. This is an important and relevant example per se, but if we change the notation and send the pair p,x into the pair x_1,x_2, and $\hbar \to \theta$, we are then considering the standard, canonical, noncommutative geometry discussed in most of this book.

The barest minimum for a manifold to be seen as a phase space is the presence of the *Poisson bracket*, a bilinear map among $C^\infty(M)$ functions on M

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M) \,, \tag{6.36}$$

with the properties of being antisymmetric, satisfying the Jacobi identity, and the Leibniz rule

$$\{f,gh\} = g\{f,h\} + \{f,g\}h \,. \tag{6.37}$$

A Poisson bracket is defined by a Poisson bivector $\Lambda \in \Gamma(M, \wedge^2 TM)$, which satisfies the (Jacobi) property

$$\Lambda^{il}\partial_l \Lambda^{jk} + \Lambda^{jl}\partial_l \Lambda^{ki} + \Lambda^{kl}\partial_l \Lambda^{ij} = 0, \tag{6.38}$$

where $\partial_i := \partial/\partial u^i$ and the u's are the local coordinates of M.

We consider the case of a two-dimensional phase space $M = \mathbb{R}^2$ with global coordinates (x,p) and Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} \,. \tag{6.39}$$

A state of the physical system is a point of the phase space, or more generally a probability distribution. The terminology is the same as in Sect. 6.1, and indeed the pure states are the points, while the non-pure states are probability distributions, in which the system is in a probabilistic superposition of states.[2] Classical observables are (real) functions on M, and the C^*-algebra they generate carries all topological information of the phase space. Some observables are the infinitesimal generators of a physically relevant transformation, the infinitesimal variation being given by the Poisson bracket, for example, time evolution is generated by the Hamiltonian function

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H,f\} \,, \tag{6.40}$$

rotations are generated by the angular momentum, etc.

[2] Note that the ensuing uncertainty of measurement is not the one inherent to measurement in quantum mechanics, but it only reflects the possibility that the state of a system is not completely known.

All this has to change drastically upon quantization and the presence of an uncertainty principle. Observables are not defined anymore as functions, but as operators on a Hilbert space, and they form a noncommutative algebra. Contact with classical mechanics is via the correspondence principle, which associates to each classical observable f an element $\hat{f}$ of a noncommutative algebra, with the basic requirement that the Poisson bracket is replaced (to leading order) by the commutator:

$$\{f,g\} \longmapsto -\frac{\mathrm{i}}{\hbar}[\hat{f},\hat{g}] \,. \tag{6.41}$$

This brings about the usual commutation relation between position and momentum:

$$[\hat{x},\hat{p}] = \mathrm{i}\hbar \,. \tag{6.42}$$

As is known $\hat{x}$ and $\hat{p}$ are unbounded operators, but it is possible to exponentiate them to obtain unitary operators and use them to build a C^*-algebra. This can then be represented as an algebra of bounded operators on some Hilbert space. The most common representation is on $L^2(\mathbb{R}_x)$, the square integrable functions of position, but one could use functions of momentum. Another commonly used representation is in terms of the eigenstates of an operator with discrete spectrum, say the Hamiltonian of the harmonic oscillator. In this case the basis of the Hilbert space is countable, and the operators can be seen as infinite matrices. We will see later on in Example 6.6 how the GNS construction applies to this case.

On square integrable functions of x the operators $\hat{x}$ and $\hat{p}$ are represented as

$$\hat{x}\psi(x) = x\psi(x), \quad \hat{p}\psi(x) = -\mathrm{i}\hbar\partial_x\psi(x). \tag{6.43}$$

The association of an operator to other functions of x and p is, however, ambiguous, and moreover it is preferable to deal with bounded operators. Weyl [15] has given a well-defined map from functions into operators, this procedure was implicitly used in Appendix 1.9. We first define the operator (sometimes called the quantizer [5] in this context)

$$\hat{W}(\eta,\xi) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\xi\cdot\hat{p}+\eta\cdot\hat{x})}. \tag{6.44}$$

The correspondence is then defined as

$$f(p,x) \longmapsto \hat{\Omega}(f)(\hat{p},\hat{x}) = \int d\xi d\eta \tilde{f}(\xi,\eta)\hat{W}(\xi,\eta) \,, \tag{6.45}$$

where

$$\tilde{f}(\xi,\eta) = \int \frac{dxdp}{2\pi} f(p,x)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\eta x+\xi p)} \tag{6.46}$$

is the Fourier transform of f. If we were to forget the hat on p and x in (6.44), the expression (6.45) would look just like the expression which Fourier transforms back $\tilde{f}$ to the original function. Because of the operatorial nature of $\hat{W}$, instead it associates an operator to functions, with the property that real functions are mapped into hermitian operators. The inverse of the Weyl map is called the *Wigner map* [16]:

$$\Omega^{-1}(\hat{F})(p,x) = \int \frac{d\eta d\xi}{(2\pi)^2\hbar}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\eta x+\xi p)} \mathrm{Tr}\hat{F}\,\hat{W}(\xi,\eta). \tag{6.47}$$

The Weyl map gives a precise prescription associating an operator to any function for which a Fourier transform can be defined. It has the characteristic of mapping real functions into hermitian operators and is a vector space isomorphism between L^2 functions on phase space and Hilbert–Schmidt operators [17–19].

The correspondence between functions and operators implicitly defines a new noncommutative product [20, 21] among functions on phase spaces defined as follows:

$$f \star g = \Omega^{-1}(\hat{\Omega}(f)\hat{\Omega}(g)) \ . \tag{6.48}$$

This product called the Grönewold–Moyal, or simply Moyal, or $\star$-product, is associative but noncommutative and it reproduces the standard quantum mechanical commutation relation:

$$x \star p - p \star x = \mathrm{i}\hbar. \tag{6.49}$$

There are several integral expressions (see for example [22, 23]) for the $\star$-product, with a fairly large domain of definition. In the context of this book it is useful to see the $\star$-product as a *twisted* convolution of Fourier transforms. Given two functions f and g the Fourier transform of their product is

$$\widetilde{(f \star g)}(\xi,\eta) = \int \frac{d\xi' d\eta'}{2\pi}\, \mathrm{e}^{\,\mathrm{i}\hbar(\xi\eta'-\xi'\eta)} \tilde{f}(\xi',\eta')\tilde{g}(\xi-\xi',\eta-\eta') \ . \tag{6.50}$$

Without the exponential this expression would just give the commutative convolution product among Fourier components. The exponential breaks the symmetry between f and g and gives noncommutativity.

Another very common form of the product is the differential expansion of the product (6.50) given by

$$(f \star g)(u) := f(u) \exp\left(\frac{\mathrm{i}\hbar}{2} \overleftarrow{\partial_i} \Lambda^{ij} \overrightarrow{\partial_j}\right) g(u) \ , \tag{6.51}$$

where the notation $\overleftarrow{\partial_i}$ (resp. $\overrightarrow{\partial_i}$) means that the partial derivative acts on the left (resp. right). This expression is an asymptotic expansion of the integral one [24], obtained by expanding the exponential in (6.50). The product can be seen as acting with the twist operator

$$\mathscr{F} = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}(\partial_x \otimes \partial_p - \partial_p \otimes \partial_x)} \tag{6.52}$$

on the tensor product of the two functions, before evaluating them on the same point. In this sense, as is discussed at length in this book, the $\star$-product is a twisted product.

Expressions (6.51) and (6.50) have different domains of definition, but they are both well defined if both function are Schwarzian functions, and in this case their product is still Schwarzian. The star product (both in the differential and integral forms) is also well defined on polynomials, which however do not belong to the C^*-

algebra, and in fact they are not mapped into bounded operators. It is nevertheless useful to consider them, which is what we do when we talk of the commutation relations (6.42). If one is not interested in the presence of the norm, then one can define the algebra of formal series in the generators x and p. This is basically the construction described in Sect. 1.2.

The asymptotic form (6.50) is convenient because it enables to write immediately the $\star$-product of two functions as a series expansion in the small parameter $\hbar$. The first term of the expansion is the ordinary commutative product. In this sense this product is a *deformation* [25] of the usual pointwise product. The second term in the expansion is proportional to the Poisson bracket:

$$f \star g = fg + \mathrm{i}\hbar\{f,g\} + O(\hbar^2)\ . \tag{6.53}$$

Considering less trivial phase spaces, starting from the work of [26, 27] a whole theory of deformed products with the property that to first order in $\hbar$ they reproduce the Poisson bracket has been developed, under the name of $\star$-quantization or (formal) deformation quantization. This culminated in the work of Kontsevich [28] who proved that it is always possible, given a manifold with a Poisson bracket, to construct a $\star$-product that quantizes the Poisson structure. That is, such that the product is associative and whose commutator, to first order in the deformation parameter, is proportional to the Poisson bracket.

Consider the Heisenberg equation of motion for observables which do not depend explicitly on time:

$$\frac{d\hat{f}}{dt} = \mathrm{i}\,\frac{[\hat{f},\hat{H}]}{\hbar} \tag{6.54}$$

and the classical analogous in terms of the Poisson bracket

$$\frac{df}{dt} = \{f,H\}, \tag{6.55}$$

where f and H are observable and the Hamiltonian for classical system, respectively, and $\hat{f},\hat{H}$ the operators obtained with the Weyl correspondence. In terms of a deformed classical mechanics we can define

$$\frac{df}{dt} = \frac{1}{\mathrm{i}\hbar}(f \star H - H \star f) = \{f,H\} + O(\hbar^2). \tag{6.56}$$

Here we can see that the two evolutions coincide in the limit $\hbar \to 0$. In this sense classical mechanics can be seen as the classical limit of quantum mechanics. The $\star$-commutator is called the *Moyal bracket* [21] and plays the role of a quantum mechanics Poisson bracket.

Example 6.5.
The algebra of functions on the (p,x) plane with the $\star$-product is isomorphic to the algebra of operators generated by $\hat{p}$ and $\hat{x}$. For further illustration in this example, we see how the algebra with the $\star$-product as well can be seen as a (infinite) matrix algebra.

Consider first the function[3]*:*

$$\varphi_0 = 2\,\mathrm{e}^{-\frac{p^2+x^2}{2}} \ . \tag{6.57}$$

This function is a projector, that is,

$$\varphi_0 \star \varphi_0 = \varphi_0 \ . \tag{6.58}$$

It is in fact the first of a whole class of projectors, as it is the function obtained applying the Wigner map to the projection operator corresponding to the ground state of the harmonic oscillator

$$\varphi_0 = \Omega^{-1}(|0\rangle\,\langle 0|). \tag{6.59}$$

Consider then the functions

$$a = \frac{1}{\sqrt{2}}(x + \mathrm{i}\,p) \quad \bar{a} = \frac{1}{\sqrt{2}}(x - \mathrm{i}\,p) \ . \tag{6.60}$$

These two operators are easily recognized as the functions corresponding (with the Wigner map) to the usual creation and annihilation operators. They have the property that for a generic function f

$$a \star f = af + \frac{\partial f}{\partial \bar{a}} \quad f \star a = af - \frac{\partial f}{\partial \bar{a}}, \tag{6.61}$$

and analogous relations involving $\bar{a}$.

Define now the functions [22]

$$\varphi_{nm} = \frac{1}{\sqrt{2^{n+m} m! n!}} \bar{a}^n \star \varphi_0 \star a^m \ . \tag{6.62}$$

These are the functions corresponding via the Wigner map to the operators $|m\rangle\,\langle n|$ and have the property

$$\varphi_{mn} \star \varphi_{kl} = \delta_{nk} \varphi_{ml} \ , \tag{6.63}$$

which is easily proven using (6.61) and (6.58).

The φ_{mn} are a basis for the functions of p and q, or alternatively of a and $\bar{a}$:

$$f = \sum_{m,n=0}^{\infty} f_{mn} \varphi_{mn}, \tag{6.64}$$

relation (6.63) ensures that

$$(f \star g)_{mn} = \sum_{p=0}^{\infty} f_{mp} g_{pn}. \tag{6.65}$$

[3] For simplicity set $\hbar = 1$.

In this sense the deformed algebra can be seen as multiplication of (infinite) matrices. □

Example 6.6.
Example 6.3 can be immediately generalized to arbitrary size matrices and even to infinite matrices (operators on $\ell^2(\mathbb{Z})$). In fact using the matrix basis described in Example 6.5 for functions $f = \sum_{mn} f_{mn}\varphi_{mn}$ the same construction can be applied using the state

$$\phi(f) = f_{00} = \int dp\,dx\; \varphi_0 \star f \star \varphi_0 \;. \tag{6.66}$$

The ideal $\mathcal{N}_\phi$ is given by functions with $f_{0m} = 0$ and we can identify the Hilbert space with functions of the kind

$$\psi = \sum_n \psi_n \varphi_{n0} \;. \tag{6.67}$$

Upon recalling that $\varphi_{n0} = \frac{1}{\sqrt{2^n n!}}\bar{a}^n \star \varphi_0$ one recognizes the usual countable basis of the Hilbert space $L^2(\mathbb{R})$ composed of Hermite polynomials multiplied by a gaussian function. □

Example 6.7.
The noncommutative torus is a compact version of the algebra described in this section. It can be seen as a deformation of the algebra of functions on the torus in the sense of Moyal. Given a function on the torus with Fourier expansion

$$f(x) = \sum_{n_1,n_2=-\infty}^{\infty} f_{n_1 n_2}\, \mathrm{e}^{in_1 x_1}\, \mathrm{e}^{in_2 x_2}, \tag{6.68}$$

we associate to it the operator

$$\hat{f} = \sum_{n_1,n_2=-\infty}^{\infty} f_{n_1 n_2} \hat{U}_1^{n_1} \hat{U}_2^{n_2}, \tag{6.69}$$

where the operators U_i act on the Hilbert space of infinite sequences of complex numbers $c = \{c_n\}$ as

$$\left(\hat{U}_1 c\right)_n = \mathrm{e}^{in\theta} c_n \;\; ; \;\; \left(\hat{U}_2 c\right)_n = c_{n+1}. \tag{6.70}$$

It is not difficult to see that the $\hat{U}$'s satisfy the relation (6.34) and the $\star$-product defined as in (6.48) can also be expressed as

$$(f \star g)(x) = e^{i\varepsilon^{ij}\theta\partial_{\xi_i}\partial_{\eta_j}} f(\xi) g(\eta)\Big|_{\xi=\eta=x}. \tag{6.71}$$

□

6.4 Final remarks

Noncommutative geometry started from the need to describe quantum mechanics, and it has led to see it as a deformation of classical mechanics. The freedom from the need to describe spaces as sets of points opened a whole new quantum world, needed on physical grounds to describe atomic physics. This deformation was the main stimulus for large body of mathematical literature, which not only helped to clarify and develop quantum mechanics, but also led to the construction of several other "noncommutative geometries", together with their symmetries. The catalog of noncommutative spaces is already large, and still growing, and noncommutative geometry has proven to be an useful tool also to understand standard, commutative geometries.

Historically quantum mechanics started from a "cutoff", imposed by Planck to avoid the ultraviolet divergences in the calculation of the black body spectrum. It is natural to think that the tools of noncommutative geometry may help the solution of the other ultraviolet divergences that we are encountering in the search for a theory that unifies quantum mechanics and gravity. Hence the study of field theories on noncommutative spaces, which is the main object of this book.

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Chapter 7
Quantum Groups, Quantum Lie Algebras and Twists

Paolo Aschieri

In this chapter, led by examples, we introduce the notions of Hopf algebra and quantum group. We study their geometry and in particular their Lie algebra (of left invariant vector fields). The examples of the quantum $sl(2)$ Lie algebra and of the quantum (twisted) Poincaré Lie algebra $iso_\theta(3,1)$ are presented.

7.1 Introduction

Hopf algebras were initially considered more than half a century ago. New important examples, named quantum groups, were studied in the 1980s [1–5]; they arose in the study of the quantum inverse scattering method in integrable systems [6]. Quantum groups can be seen as symmetry groups of noncommutative spaces, this is one reason they have been investigated in physics and mathematical physics (noncommutative spaces arise as quantization of commutative ones). The emergence of gauge theories on noncommutative spaces in open string theory in the presence of a NS 2-form background [7] has further motivated the study of noncommutative spaces and of their symmetry properties.

We here introduce the basic concepts of quantum group and of its Lie algebra of infinitesimal transformations. We pedagogically stress the connection with the classical (commutative) case and we treat two main examples, the quantum $sl(2)$ Lie algebra and the quantum Poincaré Lie algebra.

Section 7.2 shows how commutative Hopf algebras emerge from groups. The quantum group $SL_q(2)$ is then presented and its corresponding universal enveloping algebra $U_q(sl(2))$ discussed. The relation between $SL_q(2)$ and $U_q(sl(2))$ is studied in Sect. 7.5. The quantum $sl(2)$ Lie algebra, i.e., the algebra of infinitesimal transformations, is then studied in Sect. 7.6. Similarly the geometry of Hopf algebras obtained from (abelian) twists is studied via the example of the Poincaré Lie

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algebra. We do not consider here the C^*-algebras aspect of these noncommutative geometries [8].

In the appendix for reference we review some basic algebra notions and define Hopf algebras diagrammatically. We also show that in the deformation quantization context the twists we use in this book are well defined.

One aim of this chapter is to concisely introduce and relate all three aspects of quantum groups:

- deformed algebra of functions [3, 4],
- deformed universal enveloping algebra [1–3],
- quantum Lie algebra [9].

Quantum Lie algebras encode the construction of the (bicovariant) differential calculus and geometry, most relevant for physical applications. A helpful review for the first and second aspects is [10], for quantum Lie algebras we refer to [11] and [12]. The (abelian) twist case, that is an interesting subclass, can be found in [13] and is treated also in the next chapter.

7.2 Hopf algebras from groups

Let us begin with two examples motivating the notion of Hopf algebra. Let G be a finite group, and $A = Fun(G)$ be the set of functions from G to complex numbers $\mathbb{C}$. $A = Fun(G)$ is an algebra over $\mathbb{C}$ with the usual sum and product $(f+h)(g) = f(g)+h(g)$, $(f \cdot h) = f(g)h(g)$, $(\lambda f)(g) = \lambda f(g)$, for $f,h \in Fun(G)$, $g \in G$, $\lambda \in \mathbb{C}$. The unit of this algebra is I, defined by $I(g) = 1$, $\forall g \in G$. Using the group structure of G (multiplication map, existence of unit element, and inverse element), we can introduce on $Fun(G)$ three other linear maps, the coproduct (or comultiplication) Δ, the counit ε, and the antipode (or coinverse) S:

$$\Delta(f)(g,g') \equiv f(gg'), \quad \Delta : Fun(G) \to Fun(G) \otimes Fun(G), \tag{7.1}$$

$$\varepsilon(f) \equiv f(1_G), \quad \varepsilon : Fun(G) \to \mathbb{C}, \tag{7.2}$$

$$(Sf)(g) \equiv f(g^{-1}), \quad S : Fun(G) \to Fun(G), \tag{7.3}$$

where 1_G is the unit of G.

In general a coproduct can be expanded on $Fun(G) \otimes Fun(G)$ as

$$\Delta(f) = \sum_i f_1^i \otimes f_2^i \equiv f_1 \otimes f_2, \tag{7.4}$$

where $f_1^i, f_2^i \in A = Fun(G)$ and $f_1 \otimes f_2$ is a shorthand notation we will often use in the sequel. Thus we have

$$\Delta(f)(g,g') = (f_1 \otimes f_2)(g,g') = f_1(g)f_2(g') = f(gg'). \tag{7.5}$$

It is not difficult to verify the following properties of the costructures:

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta \quad \text{(coassociativity of } \Delta\text{)}, \tag{7.6}$$
$$(id \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes id)\Delta(a) = a, \tag{7.7}$$
$$\mu(S \otimes id)\Delta(a) = \mu(id \otimes S)\Delta(a) = \varepsilon(a)I, \tag{7.8}$$

and

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(I) = I \otimes I, \tag{7.9}$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(I) = 1, \tag{7.10}$$
$$S(ab) = S(b)S(a), \quad S(I) = I, \tag{7.11}$$

where $a, b \in A = Fun(G)$ and μ is the multiplication map $\mu(a \otimes b) \equiv ab$. The product in $\Delta(a)\Delta(b)$ is the product in $A \otimes A$: $(a \otimes b)(c \otimes d) = ac \otimes bd$.

For example, the coassociativity property (7.6), $(id \otimes \Delta)\Delta(f) = (\Delta \otimes id)\Delta(f)$ reads $f_1 \otimes (f_2)_1 \otimes (f_2)_2 = (f_1)_1 \otimes (f_1)_2 \otimes f_2$, for all $f \in A$. This equality is easily seen to hold by applying it on the generic element (g, g', g'') of $G \times G \times G$ and then by using associativity of the product in G.

An algebra A (not necessarily commutative) endowed with the homomorphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{C}$, and the linear and antimultiplicative map $S : A \to A$ satisfying the properties (7.6)–(7.11) is a **Hopf algebra**. Thus $Fun(G)$ is a Hopf algebra, it encodes the information on the group structure of G.

As a second example consider now the case where G is a group of matrices, a subgroup of GL given by matrices T^a_b that satisfy some algebraic relation (for example, orthogonality conditions). We then define $A = Fun(G)$ to be the algebra of polynomials in the matrix elements T^a_b of the defining representation of G and in $\det T^{-1}$; i.e., the algebra is generated by T^a_b and $\det T^{-1}$.

Using the elements T^a_b we can write an explicit formula for the expansion (7.4) or (7.5): indeed (7.1) becomes

$$\Delta(T^a{}_b)(g, g') = T^a{}_b(gg') = T^a{}_c(g)T^c{}_b(g'), \tag{7.12}$$

since T is a matrix representation of G. Therefore,

$$\Delta(T^a{}_b) = T^a{}_c \otimes T^c{}_b. \tag{7.13}$$

Moreover, using (7.2) and (7.3), one finds

$$\varepsilon(T^a{}_b) = \delta^a_b, \tag{7.14}$$
$$S(T^a{}_b) = T^{-1\,a}{}_b. \tag{7.15}$$

Thus the algebra $A = Fun(G)$ of polynomials in the elements T^a_b and $\det T^{-1}$ is a Hopf algebra with costructures defined by (7.13)–(7.15) and (7.9)–(7.11).

The two examples presented concern commutative Hopf algebras. In the first example the information on the group G is equivalent to that on the Hopf algebra $A = Fun(G)$. We constructed A from G. In order to recover G from A notice that every element $g \in G$ can be seen as a map from A to $\mathbb{C}$ defined by $f \to f(g)$. This map is multiplicative because $fh(g) = f(g)h(g)$. The set G can be obtained from A as the set of all nonzero multiplicative linear maps from A to $\mathbb{C}$ (the set of characters of A).

Concerning the group structure of G, the product is recovered from the coproduct in A via (7.5), i.e., gg' is the new character that associates to any $f \in A$ the complex number $f_1(g)f_2(g')$. The unit of G is the character ε; the inverse g^{-1} is defined via the antipode of A.

In the second example, in order to recover the topology of G, we would need a C^*-algebra completion of the algebra $A = Fun(G)$ of polynomial functions. This can be achieved if G is compact (see for example [14]); then, up to these topological (C^*-algebra) aspects, the information concerning a matrix group G can be encoded in its commutative Hopf algebra $A = Fun(G)$. Also in the case that G is locally compact there is a notion of Hopf C^*-algebra that encodes the topology and group structure of G [8, 15].

In the spirit of noncommutative geometry we now consider noncommutative deformations $Fun_q(G)$ of the algebra $Fun(G)$. The space of points G does not exist anymore, by noncommutative or quantum space G_q is meant the noncommutative algebra $Fun_q(G)$. We consider this algebra as an "algebra of functions on the deformed space G_q". Since G is a group then $Fun(G)$ is a Hopf algebra; the noncommutative Hopf algebra obtained by deformation of $Fun(G)$ is then usually called *Quantum group*. The term quantum stems for the fact that the deformation is obtained by quantizing a Poisson (symplectic) structure of the algebra $Fun(G)$ [1, 2].

7.3 Quantum groups and $SL_q(2)$

Following [5] we consider quantum groups defined as the associative algebras A freely generated by noncommuting matrix entries T^a_b satisfying the relation

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d = T^b{}_fT^a{}_eR^{ef}{}_{cd} \tag{7.16}$$

and some other conditions depending on which classical group we are deforming (see later). The matrix R controls the noncommutativity of the T^a_b, and its elements depend continuously on a (in general complex) parameter q, or even a set of parameters. For $q \to 1$, the so-called "classical limit", we have

$$R^{ab}{}_{cd} \xrightarrow{q\to 1} \delta^a_c \delta^b_d, \tag{7.17}$$

i.e., the matrix entries T^a_b commute for $q = 1$, and one recovers the ordinary $Fun(G)$. The R-matrices for the quantum group deformation of the simple Lie groups of the A,B,C,D series were given in [5].

The associativity of A leads to a consistency condition on the R-matrix, the Yang–Baxter equation:

$$R^{a_1b_1}{}_{a_2b_2} R^{a_2c_1}{}_{a_3c_2} R^{b_2c_2}{}_{b_3c_3} = R^{b_1c_1}{}_{b_2c_2} R^{a_1c_2}{}_{a_2c_3} R^{a_2b_2}{}_{a_3b_3}. \tag{7.18}$$

For simplicity we rewrite the "RTT" equation (7.16) and the Yang–Baxter equation as

$$R_{12}T_1T_2 = T_2T_1R_{12}, \tag{7.19}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{7.20}$$

where in Eq. (7.19) $R_{12} = R$, $T_1 = T \otimes 1$, $T_2 = 1 \otimes T$ (here 1 denotes the diagonal matrix with the unit element $I \in A$ on the diagonal), while in Eq. (7.20) $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, and writing $R = R^\alpha \otimes R_\alpha$ (sum over α understood), we have $R_{13} = R^\alpha \otimes 1 \otimes R_\alpha$. Thus, for example, the entries of the matrix product $R_{12}T_1$ are $(R_{12}T_1)^{ab}{}_{cd} = R^{ab}{}_{ef}T^e{}_c\delta^f{}_d = R^{ab}{}_{ed}T^e{}_c$; we see that the repeated subscripts (like 1 in this example) mean matrix multiplication.

The Yang–Baxter equation (7.20) is a condition sufficient for the consistency of the RTT equation (7.19). Indeed the product of three distinct elements $T^a{}_b$, $T^c{}_d$, and $T^e{}_f$, indicated by $T_1T_2T_3$, can be reordered as $T_3T_2T_1$ via two different paths

$$\begin{tikzcd}[column sep=small] & T_1T_3T_2 \arrow[r] & T_3T_1T_2 \arrow[dr] & \\ T_1T_2T_3 \arrow[ur] \arrow[dr] & & & T_3T_2T_1 \\ & T_2T_1T_3 \arrow[r] & T_2T_3T_1 \arrow[ur] & \end{tikzcd} \tag{7.21}$$

by repeated use of the RTT equation. The relation (7.20) ensures that the two paths lead to the same result.

The algebra A ("the quantum group") is a noncommutative Hopf algebra whose costructures are the same as those defined for the commutative Hopf algebra $Fun(G)$ of the previous section, Eqs. (7.13)–(7.15), (7.9)–(7.11).

Note 7.1 Define $\hat{R}^{ab}{}_{cd} = R^{ba}{}_{cd}$. Then the Yang–Baxter equation becomes the braid relation

$$\hat{R}_{23}\hat{R}_{12}\hat{R}_{23} = \hat{R}_{12}\hat{R}_{23}\hat{R}_{12}\ . \tag{7.22}$$

If $\hat{R}$ satisfies $\hat{R}^2 = id$ we have that $\hat{R}$ is a representation of the permutation group. In the more general case $\hat{R}$ is a representation of the braid group. The $\hat{R}$-matrix can be used to construct invariants of knots [16] (see also [17, 18]).

Let us give the example of the quantum group $SL_q(2) \equiv Fun_q(SL(2))$, the algebra freely generated by the elements α, β, γ, and δ of the 2×2 matrix

$$T^a{}_b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{7.23}$$

satisfying the commutations

$$\begin{aligned} &\alpha\beta = q\beta\alpha, \;\; \alpha\gamma = q\gamma\alpha, \;\; \beta\delta = q\delta\beta, \;\; \gamma\delta = q\delta\gamma \\ &\beta\gamma = \gamma\beta, \;\; \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma, \quad q \in \mathbb{C} \end{aligned} \tag{7.24}$$

and

$$\det_q T \equiv \alpha\delta - q\beta\gamma = I. \tag{7.25}$$

The commutations (7.24) can be obtained from (7.16) via the R-matrix

$$R^{ab}{}_{cd} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{7.26}$$

where the rows and columns are numbered in the order 11, 12, 21, 22.

It is easy to verify that the "quantum determinant" defined in (7.25) commutes with α, β, γ, and δ, so that the requirement $\det_q T = I$ is consistent. The matrix inverse of T^a_b is

$$T^{-1a}{}_b = (\det_q T)^{-1} \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}. \tag{7.27}$$

The coproduct, counit, and coinverse of α, β, γ, and δ are determined via formulas (7.13)–(7.15) to be

$$\begin{aligned} &\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, && \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \\ &\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, && \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta, \end{aligned} \tag{7.28}$$

$$\varepsilon(\alpha) = \varepsilon(\delta) = 1, \qquad \varepsilon(\beta) = \varepsilon(\gamma) = 0, \tag{7.29}$$

$$S(\alpha) = \delta, \;\; S(\beta) = -q^{-1}\beta, \;\; S(\gamma) = -q\gamma, \;\; S(\delta) = \alpha. \tag{7.30}$$

Note 7.2 The commutations (7.24) are compatible with the coproduct Δ, in the sense that $\Delta(\alpha\beta) = q\Delta(\beta\alpha)$ and so on. In general we must have

$$\Delta(R_{12}T_1T_2) = \Delta(T_2T_1R_{12}), \tag{7.31}$$

which is easily verified using $\Delta(R_{12}T_1T_2) = R_{12}\Delta(T_1)\Delta(T_2)$ and $\Delta(T_1) = T_1 \otimes T_1$. This is equivalent to proving that the matrix elements of the matrix product $T_1 T'_1$, where T' is a matrix [satisfying (7.16)] whose elements *commute* with those of T^a_b, still obey the commutations (7.19).

Note 7.3 $\Delta(\det_q T) = \det_q T \otimes \det_q T$ so that the coproduct property $\Delta(I) = I \otimes I$ is compatible with $\det_q T = I$.

Note 7.4 The condition (7.25) can be relaxed. Then we have to include the central element $\zeta = (\det_q T)^{-1}$, so as to be able to define the inverse of the q-matrix T^a_b as in (7.27) and the coinverse of the element T^a_b as in (7.15). The q-group is then $GL_q(2)$. The reader can deduce the costructures on ζ: $\Delta(\zeta) = \zeta \otimes \zeta,\ \varepsilon(\zeta) = 1,\ S(\zeta) = \det_q T$.

7.4 Universal enveloping algebras and $U_q(sl(2))$

Another example of Hopf algebra is given by any ordinary Lie algebra g, or more precisely by the universal enveloping algebra $U(g)$ of a Lie algebra g, i.e., the algebra, with unit I, of polynomials in the generators χ_i modulo the commutation relations

$$[\chi_i, \chi_j] = C_{ij}{}^k \chi_k \,. \tag{7.32}$$

Here we define the costructures on the generators as

$$\Delta(\chi_i) = \chi_i \otimes I + I \otimes \chi_i \quad \Delta(I) = I \otimes I \tag{7.33}$$

$$\varepsilon(\chi_i) = 0 \qquad \varepsilon(I) = 1 \tag{7.34}$$

$$S(\chi_i) = -\chi_i \qquad S(I) = I \tag{7.35}$$

and extend them to all $U(g)$ by requiring Δ and ε to be linear and multiplicative, $\Delta(\chi\chi') = \Delta(\chi)\Delta(\chi')$, $\varepsilon(\chi\chi') = \varepsilon(\chi)\varepsilon(\chi')$, while S is linear and antimultiplicative. In order to show that the construction of the Hopf algebra $U(g)$ is well given, we have to check that the maps Δ, ε, S are well defined. We give the proof for the coproduct. Since $[\chi, \chi']$ is linear in the generators we have

$$\Delta[\chi, \chi'] = [\chi, \chi'] \otimes I + I \otimes [\chi, \chi'] \,, \tag{7.36}$$

on the other hand, using that Δ is multiplicative we have

$$\Delta[\chi, \chi'] = \Delta(\chi)\Delta(\chi') - \Delta(\chi')\Delta(\chi). \tag{7.37}$$

It is easy to see that these two expressions coincide.

The Hopf algebra $U(g)$ is noncommutative but it is cocommutative, i.e., for all $\zeta \in U(g)$, $\zeta_1 \otimes \zeta_2 = \zeta_2 \otimes \zeta_1$, where we used the notation $\Delta(\zeta) = \zeta_1 \otimes \zeta_2$. We have discussed deformations of commutative Hopf algebras, of the kind $A = Fun(G)$, and we will see that these are related to deformations of cocommutative Hopf algebras of the kind $U(g)$ where g is the Lie algebra of G.

We here introduce the basic example of deformed universal enveloping algebra: $U_q(sl(2))$ [1–3], which is a deformation of the usual enveloping algebra of $sl(2)$,

$$[X^+, X^-] = H \;, \quad [H, X^\pm] = 2X^\pm \,. \tag{7.38}$$

The Hopf algebra $U_q(sl(2))$ is generated by the elements K_+, K_-, X_+, and X_- and the unit element I, that satisfy the relations,

$$[X_+, X_-] = \frac{K_+^2 - K_-^2}{q - q^{-1}} \;, \tag{7.39}$$

$$K_+ X_\pm K_- = q^{\pm 1} X_\pm \;, \tag{7.40}$$

$$K_+ K_- = K_- K_+ = I \;. \tag{7.41}$$

The parameter q that appears in the right-hand side of the first two equations is a complex number. It can be checked that the algebra $U_q(sl(2))$ becomes a Hopf algebra by defining the following costructures:

$$\Delta(X_\pm) = X^\pm \otimes K_+ + K_- \otimes X_\pm \;, \quad \Delta(K_\pm) = K_\pm \otimes K_\pm, \tag{7.42}$$

$$\varepsilon(X_\pm) = 0 \;, \qquad \varepsilon(K_\pm) = 1, \tag{7.43}$$

$$S(X_\pm) = -q^{\pm 1} X_\pm \;, \qquad S(K_\pm) = K_\mp. \tag{7.44}$$

If we set $q = e^h$ and $K_+ = e^{hH/2}$, then we see that in the limit $q \to 1$ we recover the undeformed $U(sl(2))$ Hopf algebra.

The Hopf algebra $U_q(sl(2))$ is not cocommutative; however, the noncocommutativity is under control. If we consider h a formal parameter and allow in $U_q(sl(2))$ formal power series in h (as we do when we consider $e^{hH/2}$) then there exists an element $\mathscr{R}$ of $U_q(sl(2)) \otimes U_q(sl(2))$, called universal R-matrix, that governs the noncocommutativity of the coproduct Δ,

$$\sigma \Delta(\zeta) = \mathscr{R} \Delta(\zeta) \mathscr{R}^{-1} \;, \tag{7.45}$$

where σ is the flip operation, $\sigma(\zeta \otimes \xi) = \xi \otimes \zeta$. The element $\mathscr{R}$ explicitly reads

$$\mathscr{R} = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]!} (q^{H/2} X_+ \otimes q^{-H/2} X_-)^n q^{n(n-1)/2}, \tag{7.46}$$

where $[n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}}$, and $[n]! = [n][n-1] \ldots 1$.

The universal $\mathscr{R}$-matrix has two further properties

$$(\Delta \otimes id)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{23}, \quad (id \otimes \Delta)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{12}, \tag{7.47}$$

where we used the notation $\mathscr{R}_{12} = \mathscr{R} \otimes I$, $\mathscr{R}_{23} = I \otimes \mathscr{R}$, and $\mathscr{R}_{13} = \mathscr{R}^\alpha \otimes I \otimes \mathscr{R}_\alpha$, where $\mathscr{R} = \mathscr{R}^\alpha \otimes \mathscr{R}_\alpha$ (sum over α understood).

A Hopf algebra with an invertible $\mathscr{R}$-matrix that satisfies (7.45) and (7.47) is called a quasitriangular Hopf algebra. If in addition $\mathscr{R}^{-1} = \mathscr{R}_{21}$ then we have a triangular Hopf algebra. From the invertibility of $\mathscr{R}$ and (7.45) and (7.47) it can be shown that $\mathscr{R}$ satisfies the Yang–Baxter equation

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12} \;. \tag{7.48}$$

7.5 Duality

Consider a finite-dimensional Hopf algebra A, the vector space A' dual to A is also a Hopf algebra with the following product, unit, and costructures [we use the notation $\psi(a) = \langle \psi, a \rangle$ in order to stress the duality between A' and A]: $\forall \psi, \phi \in A'$, $\forall a, b \in A$

$$\langle \psi\phi, a \rangle = \langle \psi \otimes \phi, \Delta a \rangle \ , \ \ \langle I, a \rangle = \varepsilon(a), \tag{7.49}$$

$$\langle \Delta(\psi), a \otimes b \rangle = \langle \psi, ab \rangle \ , \ \ \varepsilon(\psi) = \langle \psi, I \rangle, \tag{7.50}$$

$$\langle S(\psi), a \rangle = \langle \psi, S(a) \rangle, \tag{7.51}$$

where $\langle \psi \otimes \phi \ , \ a \otimes b \rangle \equiv \langle \psi, a \rangle \langle \phi, b \rangle$. Obviously $(A')' = A$ and A and A' are dual Hopf algebras.

Consider for example the group algebra $\mathbb{C}[G]$. As vector space it is the linear span over $\mathbb{C}$ of the group elements $g \in G$, each element being by definition linearly independent. The product is the product in G extended by linearity to all $\mathbb{C}[G]$. The group algebra $\mathbb{C}[G]$ is a Hopf algebra, the coproduct, counit, and antipode are defined for all $g \in G$ as

$$\Delta(g) = g \otimes g \ , \ \ \varepsilon(g) = 1 \ , \ \ S(g) = g^{-1} \tag{7.52}$$

and extended as linear maps to all of $\mathbb{C}[G]$. When G is a finite group then $\mathbb{C}[G]$ is a finite-dimensional Hopf algebra; its dual is $Fun(G)$. Indeed any complex-valued function $f \in Fun(G)$ is extended by linearity to a complex-valued function on $\mathbb{C}[G]$. Relations (7.49)–(7.51) are easily seen to hold for the group elements of $\mathbb{C}[G]$ and extend by linearity to all $\mathbb{C}[G]$. In the case that G is a finite abelian group then this duality encompasses Pontryagin duality between G and the dual group $\hat{G}$ of one-dimensional representations of G (see for example [21]).

In the infinite-dimensional case the definition of duality between Hopf algebras is more delicate because the space of linear maps A' does not naturally inherit a coproduct, indeed the space $A' \otimes A'$ (of finite sums $\sum_{i=1}^{n} \psi_i \otimes \phi_i$) does not coincide anymore with $(A \otimes A)'$. We therefore use the notion of pairing: two Hopf algebras A and U are paired if there exists a bilinear map $\langle \ , \ \rangle \ : U \otimes A \to \mathbb{C}$ satisfying (7.49) and (7.50), (then (7.51) can be shown to follow as well). We implicitly also always assume that the pairing is non-degenerate, i.e.,

$$\forall \ \psi \in U \ \ \langle \psi, a \rangle = 0 \Rightarrow a = 0 \tag{7.53}$$

and

$$\forall \ a \in A \ \ \langle \psi, a \rangle = 0 \Rightarrow \psi = 0 \ . \tag{7.54}$$

Condition (7.53) states that U separates the points (elements) of A and vice versa for (7.54). If U and A are finite dimensional then (7.53) and (7.54) are equivalent to $A' = U$; indeed (7.53) induces the injection $a \to \langle \ , a \rangle$ of A in U', similarly, by

(7.54) $U \subseteq A'$ and therefore $A' = U$. Frequently in the literature Hopf algebras that are paired are called dual. We follow this slight abuse of language.

The Hopf algebras $Fun(G)$ and $U(g)$ described in Sects. 7.3 and 7.4 are paired if g is the Lie algebra of G. Indeed we realize g as left invariant vector fields on the group manifold. Then the pairing is defined by

$$\forall t \in g, \forall f \in Fun(G), \;\; \langle t, f \rangle = t(f)|_{1_G} \;,$$

where 1_G is the unit of G, and more in general by[1]

$$\forall tt'...t'' \in U(g), \forall f \in Fun(G), \;\; \langle tt'...t'', f \rangle = t(t'...(t''(f)))|_{1_G} \;.$$

The pairing between the Hopf algebras $Fun(SL(2))$ and $U(sl(2))$ holds also in the deformed case, so that the quantum group $SL_q(2)$ is dual to $U_q(sl(2))$. In order to show this duality we introduce a subalgebra (with generators $L^{\pm}$) of the algebra of linear maps from $Fun_q(SL(2))$ to $\mathbb{C}$. We then see that this subalgebra has a natural Hopf algebra structure dual to $SL_q(2) = Fun_q(SL(2))$. Finally we see in formula (7.75) that this subalgebra is just $U_q(sl(2))$. This duality is important because it allows to consider the elements of $U_q(sl(2))$ as (left invariant) differential operators on $SL_2(2)$. This is the first step for the construction of a differential calculus on the quantum group $SL_q(2)$.

The $L^{\pm}$ functionals
The linear functionals ${L^{\pm a}}_b$ are defined by their value on the elements T^a_b:

$$ {L^{\pm a}}_b({T^c}_d) = \langle {L^{\pm a}}_b, {T^c}_d \rangle = {R^{\pm ac}}_{bd}, \tag{7.55}$$

where

$$(R^+)^{ac}_{\;\;bd} \equiv q^{-1/2} R^{ca}_{\;\;db}, \tag{7.56}$$

$$(R^-)^{ac}_{\;\;bd} \equiv q^{1/2} (R^{-1})^{ac}_{\;\;bd}. \tag{7.57}$$

The inverse matrix R^{-1} is defined by

[1] In order to see that relations (7.49), (7.50) hold, we recall that t is left invariant if $TL_g(t|_{1_G}) = t|_g$, where TL_g is the tangent map induced by the left multiplication of the group on itself: $L_g g' = gg'$. We then have

$$t(f)|_g = \left(TL_g t|_{1_G}\right)(f) = t[f(g\tilde{g})]|_{\tilde{g}=1_G} = t[f_1(g) f_2(\tilde{g})]|_{\tilde{g}=1_G} = f_1(g) t(f_2)|_{1_G}$$

and therefore

$$\langle \tilde{t} t, f \rangle = \tilde{t}(t(f))|_{1_G} = \tilde{t} f_1|_{1_G} t f_2|_{1_G} = \langle \tilde{t} \otimes t, \Delta f \rangle \;,$$

and

$$\langle t, fh \rangle = t(f)|_{1_G} h|_{1_G} + f|_{1_G} t(h)|_{1_G} = \langle \Delta(t), f \otimes h \rangle \;.$$

$$R^{-1}{}^{ab}{}_{cd} R^{cd}{}_{ef} \equiv \delta^a_e \delta^b_f \equiv R^{ab}{}_{cd} R^{-1}{}^{cd}{}_{ef}. \tag{7.58}$$

To extend the definition (7.55) to the whole algebra A we set

$$L^{\pm a}{}_b(ab) = L^{\pm a}{}_g(a) L^{\pm g}{}_b(b), \quad \forall a, b \in A, \tag{7.59}$$

so that, for example,

$$L^{\pm a}{}_b(T^c{}_d T^e{}_f) = R^{\pm ac}{}_{gd} R^{\pm ge}{}_{bf}. \tag{7.60}$$

In general, using the compact notation introduced in Sect. 2,

$$L_1^{\pm}(T_2 T_3 \ldots T_n) = R_{12}^{\pm} R_{13}^{\pm} \ldots R_{1n}^{\pm}. \tag{7.61}$$

As is easily seen from (7.60), the Yang–Baxter equation (7.20) is a necessary and sufficient condition for the compatibility of (7.55) and (7.59) with the RTT-relations: $L_1^{\pm}(R_{23} T_2 T_3 - T_3 T_2 R_{23}) = 0$.

Finally, the value of $L^{\pm}$ on the unit I is defined by

$$L^{\pm a}{}_b(I) = \delta^a_b. \tag{7.62}$$

It is not difficult to find the commutations between $L^{\pm a}{}_b$ and $L^{\pm c}{}_d$:

$$R_{12} L_2^{\pm} L_1^{\pm} = L_1^{\pm} L_2^{\pm} R_{12}, \tag{7.63}$$

$$R_{12} L_2^{+} L_1^{-} = L_1^{-} L_2^{+} R_{12}, \tag{7.64}$$

where the product $L_2^{\pm} L_1^{\pm}$ is by definition obtained by duality from the coproduct in A, for all $a \in A$,

$$L_2^{\pm} L_1^{\pm}(a) \equiv (L_2^{\pm} \otimes L_1^{\pm}) \Delta(a) \; .$$

For example, consider

$$R_{12}(L_2^+ L_1^+)(T_3) = R_{12}(L_2^+ \otimes L_1^+)\Delta(T_3) = R_{12}(L_2^+ \otimes L_1^+)(T_3 \otimes T_3) = q R_{12} R_{32} R_{31} \tag{7.65}$$

and

$$L_1^+ L_2^+(T_3) R_{12} = q R_{31} R_{32} R_{12}, \tag{7.66}$$

so that Eq. (7.63) is proven for L^+ by virtue of the Yang–Baxter equation (7.18), where the indices have been renamed $2 \to 1, 3 \to 2, 1 \to 3$. Similarly, one proves the remaining “RLL” relations.

Note 7.5 As mentioned in [5], L^+ is upper triangular, L^- is lower triangular (this is due to the upper and lower triangularity of R^+ and R^-, respectively). From (7.63) and (7.64) we have

$$L^{\pm a}{}_a L^{\pm b}{}_b = L^{\pm b}{}_b L^{\pm b}{}_b \; ; \;\; L^{+a}{}_a L^{-b}{}_b = L^{-b}{}_b L^{+a}{}_a = \varepsilon \; . \tag{7.67}$$

We also have

$$L^{\pm 1}{}_1 L^{\pm 2}{}_2 = \varepsilon \; . \tag{7.68}$$

The algebra of polynomials in the $L^{\pm}$ functionals becomes a Hopf algebra paired to $SL_q(2)$ by defining the costructures via the duality (7.55):

$$\Delta(L^{\pm a}{}_b)(T^c{}_d \otimes T^e{}_f) \equiv L^{\pm a}{}_b(T^c{}_d T^e{}_f) = L^{\pm a}{}_g(T^c{}_d) L^{\pm g}{}_b(T^e{}_f), \tag{7.69}$$

$$\varepsilon(L^{\pm a}{}_b) \equiv L^{\pm a}{}_b(I), \tag{7.70}$$

$$S(L^{\pm a}{}_b)(T^c{}_d) \equiv L^{\pm a}{}_b(S(T^c{}_d)), \tag{7.71}$$

cf. (7.50), (7.51), so that

$$\Delta(L^{\pm a}{}_b) = L^{\pm a}{}_g \otimes L^{\pm g}{}_b, \tag{7.72}$$
$$\varepsilon(L^{\pm a}{}_b) = \delta^a_b, \tag{7.73}$$
$$S(L^{\pm a}{}_b) = L^{\pm a}{}_b \circ S\ . \tag{7.74}$$

This Hopf algebra is $U_q(sl(2))$ because it can be checked that relations (7.63), (7.64), (7.67), and (7.68) fully characterize the $L^{\pm}$ functionals, so that the algebra of polynomials in the *symbols* $L^{\pm a}{}_b$ that satisfy the relations (7.63), (7.64), (7.67), and (7.68) is isomorphic to the algebra generated by the $L^{\pm}$ *functionals* on $U_q(sl(2))$. An explicit relation between the $L^{\pm}$ matrices and the generators $X^{\pm}$ and $K^{\pm}$ of $U_q(sl(2))$ introduced in the previous section is obtained by comparing the "RLL" commutation relations with the $U_q(sl(2))$ Lie algebra relations; we obtain

$$L^+ = \begin{pmatrix} K_- & q^{-1/2}(q-q^{-1})X_+ \\ 0 & K_+ \end{pmatrix} \quad , \quad L^- = \begin{pmatrix} K_+ & 0 \\ q^{1/2}(q^{-1}-q)X_- & K_- \end{pmatrix} . \tag{7.75}$$

7.6 Quantum Lie algebra

We now turn our attention to the issue of determining the Lie algebra of the quantum group $SL_q(2)$, or equivalently the quantum Lie algebra of the universal enveloping algebra $U_q(sl(2))$.

In the undeformed case the Lie algebra of a universal enveloping algebra U (for example, $U(sl(2))$) is the unique linear subspace g of U of primitive elements, i.e., of elements χ that have coproduct (I denotes the unit in the algebra):

$$\Delta(\chi) = \chi \otimes I + I \otimes \chi\ . \tag{7.76}$$

Of course g generates U and g is closed under the usual commutator bracket $[\ ,\ ]$,

$$[u,v] = uu - vu \in g \quad \text{for all } u,v \in g\ . \tag{7.77}$$

The geometric meaning of the bracket $[u,v]$ is that it is the adjoint action of g on g,

$$[u,v] = ad_u\, v, \tag{7.78}$$

$$ad_u\, v := u_1 v S(u_2), \tag{7.79}$$

where we have used the notation $\Delta(u) = \sum_\alpha u_{1_\alpha} \otimes u_{2_\alpha} = u_1 \otimes u_2$, so that a sum over α is understood. Recalling that $\Delta(u) = u \otimes I + I \otimes u$ and that $S(u) = -u$, from (7.79) we immediately obtain (7.78). In other words, the commutator $[u,v]$ is the Lie derivative of the left invariant vector field u on the left invariant vector field v. More in general the adjoint action of U on U is given by

$$ad_\xi\, \zeta = \xi_1 \zeta S(\xi_2)\ , \tag{7.80}$$

where we used the notation (sum understood) $\Delta(\xi) = \xi_1 \otimes \xi_2$.

In the deformed case the coproduct is no more cocommutative and we cannot identify the Lie algebra of a deformed universal enveloping algebra U_q with the primitive elements of U_q, they are too few to generate U_q. We then have to relax this requirement. There are three natural conditions that according to [9] the q-Lie algebra of a q-deformed universal enveloping algebra U_q has to satisfy, see [12, 19] and [20], p. 41. It has to be a linear subspace g_q of U_q such that

$$i)\ \ g_q \text{ generates } U_q\ , \tag{7.81}$$

$$ii)\ \ \Delta(g_q) \subset g_q \otimes I + U_q(sl(2))_q \otimes g_q\ , \tag{7.82}$$

$$iii)\ \ [g_q, g_q] \subset g_q\ . \tag{7.83}$$

Here now Δ is the coproduct of U_q and $[\ ,\ ]$ denotes the q-bracket

$$[u,v] = ad_u\, v = u_1 v S(u_2), \tag{7.84}$$

where we have used the coproduct notation $\Delta(u) = u_1 \otimes u_2$. Property $iii)$ is the closure of g_q under the adjoint action. Property $ii)$ implies a minimal deformation of the Leibniz rule.

From these conditions, that do not in general single out a unique subspace g_q, it follows that the bracket $[u,v]$ is quadratic in u and v, that it has a deformed antisymmetry property, and that it satisfies a deformed Jacobi identity.

In the example $U_q = U_q(sl(2))$ we have that a quantum $sl(2)$ Lie algebra is spanned by the four linearly independent elements

$$\chi^{c_1}{}_{c_2} = \frac{1}{q-q^{-1}}[L^{+c_1}{}_b S(L^{-b}{}_{c_2}) - \delta^{c_1}_{c_2} I]\ . \tag{7.85}$$

In the commutative limit $q \to 1$, we have $\chi^2_{\ 2} = -\chi^1_{\ 1} = H/2$, $\chi^1_{\ 2} = X_+$, $\chi^2_{\ 1} = X_-$ and we recover the usual $sl(2)$ Lie algebra.

The q-Lie algebra commutation relations explicitly are

$$\chi_1\chi_+ - \chi_+\chi_1 + (q^4-q^2)\chi_+\chi_2 = q^3\chi_+,$$
$$\chi_1\chi_- - \chi_-\chi_1 - (q^2-1)\chi_-\chi_2 = -q\chi_-,$$
$$\chi_1\chi_2 - \chi_2\chi_1 = 0,$$
$$\chi_+\chi_- - \chi_-\chi_+ - (1-q^2)\chi_1\chi_2 + (1-q^2)\chi_2\chi_2 = q(\chi_1-\chi_2),$$
$$\chi_2\chi_+ - q^2\chi_+\chi_2 = -q\chi_+,$$
$$\chi_2\chi_- - q^{-2}\chi_-\chi_2 = q^{-1}\chi_-,$$

where we used the composite index notation

$${}^{a_1}{}_{a_2} \to {}_i\ ,\quad {}_{b_1}{}^{b_2} \to {}^j \quad \text{and} \quad i,j=1,+,-,2\ .$$

These q-Lie algebra relations can be compactly written [12][2]

$$[\chi_i,\chi_j] = \chi_i\chi_j - \Lambda^{rs}{}_{ji}\chi_s\chi_r\ , \tag{7.86}$$

where $\Lambda^{a_2}_{a_1}{}^{d_2}_{d_1}{}^{c_1}_{c_2}{}^{b_1}_{b_2} = S(L^{+b_1}{}_{a_1})L^{-a_2}{}_{b_2}(T^{c_1}{}_{d_1}S(T^{d_2}{}_{c_2}))$. The q-Jacobi identities then read

$$[\chi_i,[\chi_j,\chi_r]] = [[\chi_i,\chi_j],\chi_r] + \Lambda^{kl}{}_{ji}[\chi_l,[\chi_k,\chi_r]]\ . \tag{7.87}$$

7.7 Deformation by twist and quantum Poincaré Lie algebra

In this last section, led by the example of the Poincaré Lie algebra, we review a quite general method to deform the Hopf algebra $U(g)$, the universal enveloping algebra of a given Lie algebra g. It is based on a twist procedure. A twist element $\mathscr{F}$ is an invertible element in $U(g)\otimes U(g)$. A main property $\mathscr{F}$ has to satisfy is the cocycle condition

$$(\mathscr{F}\otimes I)(\Delta\otimes id)\mathscr{F} = (I\otimes\mathscr{F})(id\otimes\Delta)\mathscr{F}\ . \tag{7.88}$$

Consider for example the usual Poincaré Lie algebra $iso(3,1)$:

$$[P_\mu,P_\nu] = 0\ ,$$
$$[P_\rho,M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)\ , \tag{7.89}$$
$$[M_{\mu\nu},M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}). \tag{7.90}$$

A twist element is given by

$$\mathscr{F} = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu}\ , \tag{7.91}$$

[2] Relation to the conventions of [9, 12] (here underlined): $\chi_i = -S^{-1}\underline{\chi}_i$, $f_j{}^i = S^{-1}\underline{f}^i{}_j$.

where $\theta^{\mu\nu}$ (despite the indices $\mu\nu$ notation) is a real antisymmetric matrix of dimensionful constants (the previous deformation parameter q was a constant too). We consider $\theta^{\mu\nu}$ fundamental physical constants, like the velocity of light c, or like $\hbar$. In this setting symmetries will leave $\theta^{\mu\nu}$, c, and $\hbar$ invariant. The inverse of $\mathscr{F}$ is

$$\mathscr{F} = e^{\frac{-i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu} \ .$$

This twist satisfies the cocycle condition (7.88) because the Lie algebra of momenta is abelian (for a proof see (8.8) and (8.9)).

The Poincaré Hopf algebra $U^{\mathscr{F}}(iso(3,1))$ is a deformation of $U(iso(3,1))$, as algebras $U^{\mathscr{F}}(iso(3,1)) = U(iso(3,1))$; but $U^{\mathscr{F}}(iso(3,1))$ has the new coproduct

$$\Delta^{\mathscr{F}}(\xi) = \mathscr{F}\Delta(\xi)\mathscr{F}^{-1} \ , \tag{7.92}$$

for all $\xi \in U(iso(3,1))$. The coassociativity property (7.6) for $\Delta^{\mathscr{F}}$ holds because of the cocycle condition (7.88) for $\mathscr{F}$ (for a proof see for example [21]). In order to write the explicit expression for $\Delta^{\mathscr{F}}(P_\mu)$ and $\Delta^{\mathscr{F}}(M_{\mu\nu})$, we use the Hadamard formula

$$Ad_{e^X}Y = e^X \ Y \ e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[X,[X,...[X}_{n},Y]]] = \sum_{n=0}^{\infty} \frac{(ad_X)^n}{n!} \ Y$$

and the relation $[P \otimes P', M \otimes 1] = [P,M] \otimes P'$, and thus obtain [22, 23]

$$\begin{aligned} \Delta^{\mathscr{F}}(P_\mu) &= P_\mu \otimes I + I \otimes P_\mu \ , \\ \Delta^{\mathscr{F}}(M_{\mu\nu}) &= M_{\mu\nu} \otimes I + I \otimes M_{\mu\nu} \\ &\quad - \frac{1}{2}\theta^{\alpha\beta}\left((\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\right) . \end{aligned} \tag{7.93}$$

We have constructed the Hopf algebra $U^{\mathscr{F}}(iso(3,1))$: it is the algebra generated by $M_{\mu\nu}$ and P_μ modulo the relations (7.89) and with coproduct (7.93) and counit and antipode that are as in the undeformed case:

$$\varepsilon(P_\mu) = \varepsilon(M_{\mu\nu}) = 0, \quad S(P_\mu) = -P_\mu, \quad S(M_{\mu\nu}) = -M_{\mu\nu} \ . \tag{7.94}$$

This algebra is a symmetry algebra of the noncommutative spacetime $\hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu = i\theta^{\mu\nu}$.

In general given a Lie algebra g, and a twist $\mathscr{F} \in U(g) \otimes U(g)$, formula (7.92) defines a new coproduct that is not cocommutative. We call $U(g)^{\mathscr{F}}$ the new Hopf algebra with coproduct $\Delta^{\mathscr{F}}$, counit $\varepsilon^{\mathscr{F}} = \varepsilon$, and antipode $S^{\mathscr{F}}$ that is a deformation of S [24, 25].[3] By definition as algebra $U(g)^{\mathscr{F}}$ equals $U(g)$, only the costructures are deformed.

[3] Explicitly, if we write $\mathscr{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha$ (sum over α understood) and define the element $\chi = \mathrm{f}^\alpha S(\mathrm{f}_\alpha)$ (that can be proven to be invertible) then for all elements $\xi \in U(g)$, $S^{\mathscr{F}}(\xi) = \chi S(\xi)\chi^{-1}$.

We now construct the quantum Poincaré Lie algebra $iso^{\mathscr{F}}(3,1)$. Following the previous section, the Poincaré Lie algebra $iso^{\mathscr{F}}(3,1)$ must be a linear subspace of $U^{\mathscr{F}}(iso(3,1))$ such that if $\{t_i\}_{i=1,\ldots,n}$ is a basis of $iso^{\mathscr{F}}(3,1)$, we have (sum understood on repeated indices)

$$\begin{aligned} i)&\quad \{t_i\} \text{ generates } U^{\mathscr{F}}(iso(3,1)) \\ ii)&\quad \Delta^{\mathscr{F}}(t_i) = t_i \otimes I + f_i{}^j \otimes t_j \\ iii)&\quad [t_i, t_j]_{\mathscr{F}} = C_{ij}{}^k t_k \end{aligned}$$

where $C_{ij}{}^k$ are structure constants and $f_i{}^j \in U^{\mathscr{F}}(iso(3,1))$ $(i,j = 1,\ldots,n)$. In the last line the bracket $[\ ,\ ]_{\mathscr{F}}$ is the adjoint action:

$$[t,t']_{\mathscr{F}} := ad_t^{\mathscr{F}}\, t' = t_{1_{\mathscr{F}}} t' S(t_{2_{\mathscr{F}}})\;, \tag{7.95}$$

where we used the coproduct notation $\Delta^{\mathscr{F}}(t) = t_{1_{\mathscr{F}}} \otimes t_{2_{\mathscr{F}}}$. The statement that the Lie algebra of $U^{\mathscr{F}}(iso(3,1))$ is the undeformed Poincaré Lie algebra (7.89) is not correct because conditions $ii)$ and $iii)$ are not met by the generators P_μ and $M_{\mu\nu}$. As we discuss in the next chapter (see in particular Sect. 8.2.3.1), there is a canonical procedure [13] in order to obtain the Lie algebra $iso^{\mathscr{F}}(3,1)$ of $U^{\mathscr{F}}(iso(3,1))$. Consider the elements

$$P_\mu^{\mathscr{F}} := \bar{\mathrm{f}}^\alpha(P_\mu)\bar{\mathrm{f}}_\alpha = P_\mu\;, \tag{7.96}$$

$$\begin{aligned} M_{\mu\nu}^{\mathscr{F}} := \bar{\mathrm{f}}^\alpha(M_{\mu\nu})\bar{\mathrm{f}}_\alpha &= M_{\mu\nu} - \frac{i}{2}\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}]P_\sigma \\ &= M_{\mu\nu} + \frac{1}{2}\theta^{\rho\sigma}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)P_\sigma. \end{aligned} \tag{7.97}$$

Their coproduct is

$$\begin{aligned} \Delta^{\mathscr{F}}(P_\mu) &= P_\mu \otimes I + I \otimes P_\mu\;, \\ \Delta^{\mathscr{F}}(M_{\mu\nu}^{\mathscr{F}}) &= M_{\mu\nu}^{\mathscr{F}} \otimes I + I \otimes M_{\mu\nu}^{\mathscr{F}} + i\theta^{\alpha\beta}P_\alpha \otimes [P_\beta, M_{\mu\nu}]\;. \end{aligned} \tag{7.98}$$

The counit and antipode are

$$\begin{aligned} &\varepsilon(P_\mu) = \varepsilon(M_{\mu\nu}^{\mathscr{F}}) = 0\;, \\ &S(P_\mu) = -P_\mu\;,\quad S(M_{\mu\nu}^{\mathscr{F}}) = -M_{\mu\nu}^{\mathscr{F}} - i\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}]P_\sigma\;. \end{aligned} \tag{7.99}$$

The elements $P_\mu^{\mathscr{F}}$ and $M_{\mu\nu}^{\mathscr{F}}$ are generators because they satisfy condition $i)$ (indeed $M_{\mu\nu} = M_{\mu\nu}^{\mathscr{F}} + \frac{i}{2}\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}^{\mathscr{F}}]P_\sigma$). They are deformed *infinitesimal* generators because they satisfy the Leibniz rule $ii)$ and because they close under the Lie bracket $iii)$. Explicitly

$$\begin{aligned} [P_\mu, P_\nu]_{\mathscr{F}} &= 0\;, \\ [P_\rho, M_{\mu\nu}^{\mathscr{F}}]_{\mathscr{F}} &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)\;, \end{aligned}$$

$$[M^{\mathscr{F}}_{\mu\nu}, M^{\mathscr{F}}_{\rho\sigma}]_{\mathscr{F}} = -i(\eta_{\mu\rho}M^{\mathscr{F}}_{\nu\sigma} - \eta_{\mu\sigma}M^{\mathscr{F}}_{\nu\rho} - \eta_{\nu\rho}M^{\mathscr{F}}_{\mu\sigma} + \eta_{\nu\sigma}M^{\mathscr{F}}_{\mu\rho}) \,. \quad (7.100)$$

We notice that the structure constants are the same as in the undeformed case; however, the adjoint action $[M^{\mathscr{F}}_{\mu\nu}, M^{\mathscr{F}}_{\rho\sigma}]_{\mathscr{F}}$ is not the commutator anymore, it is a deformed commutator quadratic in the generators and antisymmetric:

$$\begin{aligned} [P_\mu, P_\nu]_{\mathscr{F}} &= [P_\mu, P_\nu]\,, \\ [P_\rho, M^{\mathscr{F}}_{\mu\nu}]_{\mathscr{F}} &= [P_\rho, M^{\mathscr{F}}_{\mu\nu}]\,, \\ [M^{\mathscr{F}}_{\mu\nu}, M^{\mathscr{F}}_{\rho\sigma}]_{\mathscr{F}} &= [M^{\mathscr{F}}_{\mu\nu}, M^{\mathscr{F}}_{\rho\sigma}] - i\theta^{\alpha\beta}[P_\alpha, M_{\rho\sigma}][P_\beta, M_{\mu\nu}]\,. \end{aligned} \quad (7.101)$$

From (7.100) we immediately obtain the Jacobi identities:

$$[t\,, [t', t'']_{\mathscr{F}}]_{\mathscr{F}} + [t'\,, [t'', t]_{\mathscr{F}}]_{\mathscr{F}} + [t''\,, [t, t']_{\mathscr{F}}]_{\mathscr{F}} = 0\,, \quad (7.102)$$

for all $t, t', t'' \in iso^{\mathscr{F}}(3,1)$.

Appendix

7.8 Algebras, coalgebras, and Hopf algebras

In the introduction we have motivated the notion of Hopf algebra. We here review some basic definitions in linear algebra and show how Hopf algebras merge algebra and coalgebra structures in a symmetric (specular) way [26, 27], [21].

We recall that a module by definition is an abelian group. The group operation is denoted $+$ (additive notation). A vector space A over $\mathbb{C}$ (or $\mathbb{R}$) is a $\mathbb{C}$-module, i.e., there is an action $(\lambda, a) \to \lambda a$ of the group $(\mathbb{C} - \{0\}, \cdot)$ on the module A,

$$(\lambda'\lambda)a = \lambda(\lambda' a) \quad , \quad 1\,a = a, \quad (7.103)$$

and this action is compatible with the addition in A and in $\mathbb{C}$, i.e., it is compatible with the module structure of A and of $\mathbb{C}$:

$$\lambda(a + a') = \lambda a + \lambda a' \quad , \quad (\lambda + \lambda')a = \lambda a + \lambda' a\,. \quad (7.104)$$

In order to introduce the tensor product $V \otimes W$ of two vector spaces V and W, we consider the vector space $F(V,W)$ freely generated by the points of $V \times W$; a generic element of $F(V,W)$ is a *finite* sum $\sum_i \lambda_i (v_i, w_i)$ where $\lambda_i \in \mathbb{C}$, and the set of all points $(v,w) \in V \times W$ is a basis of $F(V,W)$. In $F(V,W)$ we consider the subspace $R(V,W)$ generated by the elements

$$\lambda(v,w) - (\lambda v, w), \quad \lambda(v,w) - (v, \lambda w), \quad (7.105)$$

$$(v + v', w) - (v,w) - (v', w), \quad (v, w + w') - (v,w) - (v, w')\,. \quad (7.106)$$

The quotient $F(V,W)/R(V,W)$ is the tensor product space $V \otimes W$. The equivalence class of the element (v,w) is denoted $v \otimes w$, and from the definition of $R(V,W)$ we have

$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w) \, ,$$
$$(v+v') \otimes w = v \otimes w + v' \otimes w, \ v \otimes (w+w') = v \otimes w + v \otimes w' \, .$$

From the definition of $V \otimes W$ it follows that a generic element $F \in V \otimes W$ is a finite sum (over the index α) $F = f^\alpha \otimes f_\alpha$ of elements $f^\alpha \in V$, $f_\alpha \in W$. If V and W are finite dimensional and $\dim V = m$, $\dim W = n$, then $\dim(V \otimes W) = n \cdot m$.

The tensor space $V \otimes W$ can also be defined categorically: given V, W, U vector spaces, to any map $l : V \times W \to U$ linear in V and in W, there correspond a unique map $\tilde{l} : V \otimes W \to U$, such that $l(v,w) = \tilde{l}(v \otimes w)$.

An **algebra** A over $\mathbb{C}$ with unit I is a vector space over $\mathbb{C}$ with a multiplication map that we denote $\cdot$ or μ,

$$\mu \, : A \times A \to A \tag{7.107}$$

that is $\mathbb{C}$-bilinear: $(\lambda a) \cdot (\lambda' b) = \lambda \lambda' (a \cdot b)$, that is associative and that for all a satisfies $a \cdot I = I \cdot a = a$.

These three properties can be stated diagrammatically. $\mathbb{C}$-bilinearity of the product $\mu : A \times A \to A$ is equivalently expressed as linearity of the map $\mu : A \otimes A \to A$. Associativity reads,

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"'] & A \otimes A \arrow[d, "\mu"'] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

Finally the existence of the unit I such that for all a we have $a \cdot I = I \cdot a = a$ is equivalent to the existence of a linear map

$$i : \mathbb{C} \to A \tag{7.108}$$

such that

$$\begin{tikzcd} \mathbb{C} \otimes A \arrow[r, "i \otimes id"] \arrow[d, "\simeq"'] & A \otimes A \arrow[d, "\mu"'] \\ A \arrow[r, "id"] & A \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes \mathbb{C} \arrow[r, "id \otimes i"] \arrow[d, "\simeq"'] & A \otimes A \arrow[d, "\mu"'] \\ A \arrow[r, "id"] & A \end{tikzcd}$$

where $\simeq$ denotes the canonical isomorphism between $A \otimes \mathbb{C}$ (or $\mathbb{C} \otimes A$) and A. The unit I is then recovered as $i(1) = I$.

A **coalgebra** A over $\mathbb{C}$ is a vector space with a linear map $\Delta : A \to A \otimes A$ that is coassociative,

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta \ ,$$

and a linear map $\varepsilon : A \to \mathbb{C}$, called counit that satisfies

$$(id \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes id)\Delta(a) = a \ .$$

These properties can be expressed diagrammatically by reverting the arrows of the previous diagrams:

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes id"'] \\
A \otimes A \arrow[u, "id \otimes \Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C} \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes id"'] \\
A \arrow[u, "\simeq"] & A \arrow[l, "id"] \arrow[u, "\Delta"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{C} \otimes A & A \otimes A \arrow[l, "id \otimes \varepsilon"'] \\
A \arrow[u, "\simeq"] & A \arrow[l, "id"] \arrow[u, "\Delta"']
\end{tikzcd}$$

We finally arrive at the following:

Definition A **bialgebra** A over $\mathbb{C}$ is a vector space A with an algebra structure and a coalgebra structure that are compatible, i.e.,

1) the coproduct Δ is an algebra map between the algebra A and the algebra $A \otimes A$, where the product in $A \otimes A$ is $(a \otimes b)(c \otimes d) = ac \otimes bd$,

$$\Delta(ab) = \Delta(a)\Delta(b), \ \ \Delta(I) = I \otimes I. \tag{7.109}$$

2) The counit $\varepsilon : A \to \mathbb{C}$ is an algebra map

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \ \ \varepsilon(I) = 1 \ . \tag{7.110}$$

Definition A **Hopf algebra** is a bialgebra with a linear map $S : A \to A$, called antipode (or coinverse), such that

$$\mu(S \otimes id)\Delta(a) = \mu(id \otimes S)\Delta(a) = \varepsilon(a)I \ . \tag{7.111}$$

It can be proven that the antipode S is unique and antimultiplicative

$$S(ab) = S(b)S(a) \ .$$

From the definition of bialgebra it follows that $\mu : A \otimes A \to A$ and $i : \mathbb{C} \to A$ are coalgebra maps, i.e., $\Delta \circ \mu = \mu \otimes \mu \circ \underline{\Delta}$, $\varepsilon \otimes \mu = \underline{\varepsilon}$ and $\Delta \circ i = i \otimes i \circ \Delta_{\mathbb{C}}$, $\varepsilon \circ i = \varepsilon_{\mathbb{C}}$, where the coproduct and counit in $A \otimes A$ are given by $\underline{\Delta}(a \otimes b) = a_1 \otimes b_1 \otimes a_2 \otimes b_2$ and $\underline{\varepsilon} = \varepsilon \otimes \varepsilon$, while the coproduct in $\mathbb{C}$ is the map $\Delta_{\mathbb{C}}$ that identifies $\mathbb{C}$ with $\mathbb{C} \otimes \mathbb{C}$ and the counit is $\varepsilon_{\mathbb{C}} = id$. Vice versa if A is an algebra and a coalgebra and μ and i are coalgebra maps then it follows that Δ and ε are algebra maps.

One can write diagrammatically Eqs. (7.109), (7.110), (7.111) and see that the Hopf algebra definition is invariant under inversion of arrows and exchange of structures with costructures, with the antipode going into itself. In this respect the algebra and the coalgebra structures in a Hopf algebra are dual (specular). This property implies that the space H' of linear maps of a finite-dimensional Hopf algebra H is a Hopf algebra itself (cf. Sect. 7.5).

7.9 Hopf algebra twists

Definition A **twist** of a Hopf algebra H is an element $\mathscr{F} \in H \otimes H$ that is invertible, that satisfies the cocycle condition

$$(\mathscr{F} \otimes I)(\Delta \otimes id)\mathscr{F} = (I \otimes \mathscr{F})(id \otimes \Delta)\mathscr{F} \tag{7.112}$$

and that is properly normalized, i.e.,

$$(id \otimes \varepsilon)\mathscr{F} = (\varepsilon \otimes id)\mathscr{F} = 1 \otimes 1 \ . \tag{7.113}$$

In this book we consider twists $\mathscr{F}$ of Hopf algebras Ug that are universal enveloping algebras (or of quantum universal enveloping algebras $U_q g$ as in Sect. 9.3.2). In order for these elements $\mathscr{F}$ (that are typically the exponential of elements in $g \otimes g$) to be mathematically well-defined twists, some care is needed. The aim of this section is to show that, in the deformation quantization context we use, these elements $\mathscr{F}$ are well-defined examples of Hopf algebra twists.

We have considered algebras A over the field $\mathbb{C}$. More in general we can consider algebras over a commutative ring R. We recall that a commutative ring R is a module with a map $\mu : R \times R \to R$ that is associative and commutative and compatible with the module structure, i.e., the product μ is distributive over the addition $+$; we consider rings with unit element $1 \in R$.

An algebra A over a commutative ring R is an R-module, i.e., it is a module that satisfies the properties (7.103) and (7.104) (with $\lambda \in R$ rather than $\mathbb{C}$) and with the multiplication map $\mu : A \times A \to A$ that is R-bilinear.

An example of ring is the ring $\mathbb{C}[[h]]$ of formal power series in h over $\mathbb{C}$. The universal enveloping algebra Ug of the Lie algebra g is an algebra over $\mathbb{C}$, while $Ug[[h]]$ (formal power series in h over Ug) is an algebra over $\mathbb{C}[[h]]$.

We now observe that the tensor product construction $U \otimes W$ holds also if V and W are R-modules, just consider $\lambda, \lambda' \in R$ in (7.103) and (7.104). In particular we can consider $Ug[[h]] \otimes Ug[[h]]$, where the tensor product is over $\mathbb{C}[[h]]$.

We can now show that in this context the twists $\mathscr{F}$ we consider in this book are well-defined twists because $\mathscr{F}$ are elements of $Ug[[h]] \otimes Ug[[h]]$. For example, consider the abelian Lie algebra of partial derivatives ∂_μ on Minkowski space and the twist

$$\mathscr{F} = e^{-\frac{i}{2} h \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu} .$$

The exponential is considered a formal power series expansion in h,

$$\begin{aligned} \mathscr{F} &= e^{-\frac{i}{2} h \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}} \\ &= \sum \frac{h^n}{n!} \left(-\frac{i}{2}\right)^n \theta^{\mu_1\nu_1} \ldots \theta^{\mu_n\nu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} \otimes \partial_{\nu_1} \ldots \partial_{\nu_n}, \end{aligned} \tag{7.114}$$

the coefficient of h^n is a finite sum of elements of $Ug \otimes_{\mathbb{C}} Ug$, this shows that $\mathscr{F}$ is an element of $(Ug \otimes_{\mathbb{C}} Ug)[[h]]$. Obviously $(Ug \otimes_{\mathbb{C}} Ug)[[h]] = Ug[[h]] \otimes_{\mathbb{C}[[h]]} Ug[[h]]$, and therefore $\mathscr{F}$ is indeed a twist of the Hopf algebra $Ug[[h]]$ (over the ring $\mathbb{C}[[h]]$).

Notice that on the other hand if we consider h a complex number then strictly speaking $\mathscr{F}$ does not belong to $Ug \otimes Ug$ (tensor product over $\mathbb{C}$) because the exponential gives an infinite sum of elements of Ug.

These subtleties can frequently be ignored in physical applications. There one considers Lagrangian field theories where fields are $\star$-multiplied. These theories are deformations of usual field theories. Quite a few aspects of these theories can be understood by considering a power series expansion in the noncommutativity parameter. In this case one neglects terms higher than a fixed one, say h^n, in the action functional. Then h can be considered a (possibly small) real number and $\theta^{\mu\nu}$ are dimensionful parameters responsible for new interaction terms (interaction terms in the action due to the noncommutativity of spacetime).

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Chapter 8
Noncommutative Symmetries and Gravity

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Spacetime geometry is twisted (deformed) into noncommutative spacetime geometry, where functions and tensors are now star multiplied. Consistently, spacetime diffeomorphisms are twisted into noncommutative diffeomorphisms. Their deformed Lie algebra structure and that of infinitesimal Poincaré transformations is defined and explicitly constructed. We can then define covariant derivatives (that implement the principle of general covariance on noncommutative spacetime) and torsion and curvature tensors. With these geometric tools we formulate a noncommutative theory of gravity.

8.1 Introduction

The study of the structure of spacetime at Planck scale, where quantum gravity effects are non-negligible, is a main open challenge in fundamental physics. Since the dynamical variable in Einstein general relativity is spacetime itself (with its metric structure) and since in quantum mechanics and in quantum field theory the classical dynamical variables become noncommutative, one is led to conclude that noncommutative spacetime is a feature of Planck scale physics. This expectation is further supported by Gedanken experiments that aim at probing spacetime structures at very small distances. They show that due to gravitational backreaction one cannot test spacetime at Planck scale. For example, in relativistic quantum mechanics the position of a particle can be detected with a precision at most of the order of its Compton wavelength $\lambda_C = \hbar/mc$. Probing spacetime at infinitesimal distances implies an extremely heavy particle that in turn curves spacetime itself. When λ_C is of the order of the Planck length, the spacetime curvature radius due to the particle has the same order of magnitude and the attempt to measure spacetime structure beyond Planck scale fails.

Gedanken experiments of this type support finite reductionism. They show that the description of spacetime as a continuum of points (a smooth manifold) is an

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assumption no more justified at Planck scale. It is then natural to relax this assumption and conceive a more general noncommutative spacetime, where uncertainty relations and discretization naturally arise. In this way one can argue for the impossibility of an operational definition of continuous Planck length spacetime (i.e., a definition given by describing the operations to be performed for at least measuring spacetime by a Gedanken experiment). A dynamical feature of spacetime could be incorporated at a deeper kinematical level.

As an example compare Galilean relativity to special relativity. Contraction of distances and time dilatation can be explained in Galilean relativity: they are a consequence of the interaction between ether and the body in motion. In special relativity they have become a kinematical feature.

This line of thought has been anticipated by Riemann, see Sect. 1.1, considered in [1], and more recently in [2–17] (see also the review [18]).

We also notice that uncertainty relations in position measurements are in agreement with string theory models [19–25] and that non-perturbative attempts to describe string theories have shown that a noncommutative structure of spacetime emerges [26].

A first question to be asked in the context we have outlined is whether one can consistently deform Riemannian geometry into a noncommutative Riemannian geometry. In this chapter we address this question. We construct a noncommutative version of differential and of Riemannian geometry and obtain the noncommutative version of Einstein equations.

We consider noncommutative deformations of the algebra of functions on a smooth manifold M obtained by deforming the usual pointwise product to a $\star$-product. It is possible to consider a wide class of $\star$-products. These $\star$-products are associated with a twist $\mathscr{F}$ of the Lie algebra of infinitesimal diffeomorphisms on the smooth manifold M. For pedagogical reasons in this chapter we treat mainly the case of constant noncommutativity, $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$, and we assume that commutative spacetime as a manifold is $\mathbb{R}^4$. The general case is discussed in [16] ([27]). In this case the twist $\mathscr{F}$ is arbitrary; it is tempting to introduce equations of motion for $\mathscr{F}$ and thus describe a spacetime geometry where both the metric aspect and the noncommutative aspect are dynamical.

As argued, noncommutativity should be relevant at Planck scale; however, the physical phenomena it induces can also appear at lower energies. Consider for example the perturbations responsible for structure formation and for the temperature anisotropies in the cosmic microwave background radiation. They arise as quantum fluctuations during the inflationary epoch and are stretched to cosmological scales by the exponential expansion. Hence these perturbations are sensitive to physics at distances at least as small as the horizon size during inflation which is not far from Planck scale. Noncommutativity of spacetime at inflation scale leads to quadrupole moment contributions to the cosmic microwave background spectrum [28].

Another interesting study is the formulation of the noncommutative analogue of the Friedmann–Robertson–Walker spacetime, as well as of other classical solutions of the Einstein equations.

Even without physical motivations, the mathematical structure of deformed spaces is a challenging and fruitful research arena. It is very surprising how well $\star$-noncommutative structures can be incorporated in the framework of differential geometry.

Structure of the chapter
In Sect. 8.2 we introduce the twist $\mathscr{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$. The general notion of twist is well known [29, 30]. Multiparametric twists appear in [31]. In the context of deformed Poincaré group and Minkowski space geometry twists have been studied in [32–36] (multiparametric deformations) and in [37–43] (Moyal–Weyl deformations).

Given a twist $\mathscr{F}$ we state the general principle that allows to construct noncommutative products by composing commutative products with the twist $\mathscr{F}$. In this way we obtain the algebras of noncommutative functions, tensor fields, exterior forms, and diffeomorphisms. Noncommutative diffeomorphisms are then shown to naturally act on tensor fields and forms. We study in detail the notion of infinitesimal diffeomorphism and the corresponding notion of deformed Lie algebra.

In Sect. 8.3 we present the example of the Poincaré symmetry, give explicitly the infinitesimal generators and their deformed Lie bracket, and explain the geometric origin of the latter. The generators and the bracket differ from the ones usually considered in the literature.

In Sect. 8.4 we use the noncommutative differential geometry formalism introduced in Sect. 8.2 and develop the notion of covariant derivative and of torsion, curvature, and Ricci curvature tensors.

In Sect. 8.5 a metric on noncommutative space is introduced. The corresponding unique torsion-free metric compatible connection is used to construct the Ricci tensor and obtain the Einstein equations for gravity on noncommutative spacetime.

In Appendix 8.6 we show that the algebra of differential operators is not a Hopf algebra, and we relate it to the Hopf algebra of infinitesimal diffeomorphisms.

8.2 Deformation by twists

A quite general procedure in order to construct noncommutative spaces and noncommutative field theories is that of a twist. The ingredients are

I) A Lie algebra g.

II) An action of the Lie algebra on the space one wants to deform.

III) A twist element $\mathscr{F}$, constructed with the generators of the Lie algebra g.

8.2.1 The twist $\mathscr{F}$

A twist element $\mathscr{F}$ is an invertible element in $Ug \otimes Ug$, where Ug is the universal enveloping algebra of g. Ug is a Hopf algebra, in particular there is a linear map, called coproduct

$$\Delta : Ug \to Ug \otimes Ug \ . \tag{8.1}$$

For every Lie algebra element $t \in g$ we have[1]

$$\Delta(t) = t \otimes 1 + 1 \otimes t \, . \tag{8.2}$$

The coproduct Δ is extended to all Ug by defining

$$\Delta(tt') := \Delta(t)\Delta(t') = tt' \otimes 1 + t \otimes t' + t' \otimes t + 1 \otimes tt'$$

and more generally $\Delta(tt' \ldots t'') = \Delta(t)\Delta(t') \ldots \Delta(t'')$. The main property $\mathscr{F}$ has to satisfy is the cocycle condition

$$(\mathscr{F} \otimes 1)(\Delta \otimes id)\mathscr{F} = (1 \otimes \mathscr{F})(id \otimes \Delta)\mathscr{F} \ . \tag{8.3}$$

If g is the Lie algebra of vector fields on spacetime $M = \mathbb{R}^4$, or simply the subalgebra spanned by the commuting vector fields $\partial/\partial x^\mu$, we can consider the twist

$$\mathscr{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}} \ , \tag{8.4}$$

with $\theta^{\mu\nu}$ an antisymmetric constant matrix. The inverse of $\mathscr{F}$ is

$$\mathscr{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}} \ .$$

The star product between functions can be obtained from the usual pointwise product via the action of the twist operator, namely,

$$f \star g := \mu \circ \mathscr{F}^{-1}(f \otimes g) \ , \tag{8.5}$$

where μ is the usual pointwise product between functions, $\mu(f \otimes g) = fg$.

Despite the $\mu\nu$ index notation, we will consistently consider the entries $\theta^{\mu\nu}$ of the antisymmetric matrix θ as fundamental dimensionful constants, like c or $\hbar$. In particular the deformed spacetime symmetries we consider will leave invariant the θ matrix. The point is that the exponent of $\mathscr{F}$,

$$\theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}$$

[1] In this chapter for ease of notation we denote the unit of Ug simply by 1 (and not by I as in the previous chapter).

is not the Poisson tensor associated with the $\star$-product. The difference lies in the tensor product $\otimes$. The Poisson tensor is

$$\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes_A\frac{\partial}{\partial x^\nu}, \tag{8.6}$$

where we have explicitly written that the tensor product is over the algebra $A = Fun(\mathbb{R}^4)$ of smooth functions on spacetime. On the other hand the tensor product in $\mathscr{F}$ is over the complex numbers, we should write

$$\mathscr{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes_{\mathbb{C}}\frac{\partial}{\partial x^\nu}}\ .$$

That is why $\theta^{\mu\nu}$ in $\mathscr{F}$ is not a tensor but a set of constants. In this respect, a better notation for $\mathscr{F}$ is

$$\mathscr{F} = e^{\frac{-i}{2}\theta^{ab}X_a\otimes X_b}\ , \tag{8.7}$$

where $a,b = 1,...,4$ and $X_1 = \frac{\partial}{\partial x^1}, \ldots, X_4 = \frac{\partial}{\partial x^4}$ are globally defined vector fields on spacetime.

It is easy to prove that $\mathscr{F}$ satisfies the cocycle condition (8.3). Since the coproduct Δ is multiplicative we have

$$(\Delta\otimes id)\mathscr{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\Delta(\frac{\partial}{\partial x^\mu})\otimes\frac{\partial}{\partial x^\nu}} = e^{-\frac{i}{2}\theta^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\otimes 1\otimes\frac{\partial}{\partial x^\nu}+1\otimes\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}\right)}, \tag{8.8}$$

and therefore, since partial derivatives mutually commute,

$$(\mathscr{F}\otimes 1)(\Delta\otimes id)\mathscr{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}\otimes 1+\frac{\partial}{\partial x^\mu}\otimes 1\otimes\frac{\partial}{\partial x^\nu}+1\otimes\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}\right)}\ . \tag{8.9}$$

The right-hand side of (8.3) is easily proven to coincide with this expression.

We shall frequently use the notation (sum over $\alpha = 1,2,...,\infty$ understood)

$$\mathscr{F} = \mathrm{f}^\alpha\otimes\mathrm{f}_\alpha, \quad \mathscr{F}^{-1} = \overline{\mathrm{f}}^\alpha\otimes\overline{\mathrm{f}}_\alpha\ , \tag{8.10}$$

where, for each value of α, $\overline{\mathrm{f}}^\alpha$ and $\overline{\mathrm{f}}_\alpha$ are two distinct elements of Ug (and similarly $\mathrm{f}^\alpha, \mathrm{f}_\alpha \in Ug$). Explicitly these elements are

$$\begin{aligned}\mathscr{F}^{-1} &= e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}}\\ &= \sum_n \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\partial_{\mu_1}\ldots\partial_{\mu_n}\otimes\partial_{\nu_1}\ldots\partial_{\nu_n}\\ &= \overline{\mathrm{f}}^\alpha\otimes\overline{\mathrm{f}}_\alpha\ .\end{aligned} \tag{8.11}$$

From this expression we also see that α is a multi-index, it runs over all the values of the indices $\nu, \nu_1\nu_2, \nu_1\nu_2\nu_3, \ldots$. Using this notation the $\star$-product between functions reads

$$f\star g := \overline{\mathrm{f}}^\alpha(f)\overline{\mathrm{f}}_\alpha(g)\ . \tag{8.12}$$

We also introduce the universal $\mathscr{R}$-matrix

$$\mathscr{R} := \mathscr{F}_{21}\mathscr{F}^{-1}, \tag{8.13}$$

where by definition $\mathscr{F}_{21} = \mathrm{f}_\alpha \otimes \mathrm{f}^\alpha$. In the sequel we use the notation

$$\mathscr{R} = R^\alpha \otimes R_\alpha, \qquad \mathscr{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha \ . \tag{8.14}$$

In the case of the twist (8.4) we simply have $\mathscr{R} = \mathscr{F}^{-2}$ but for more general twists this is no more the case. The $\mathscr{R}$-matrix measures the noncommutativity of the $\star$-product. Indeed it is easy to see that due to the antisymmetry of $\theta^{\mu\nu}$, $\overline{\mathrm{f}}^\alpha(h)\overline{\mathrm{f}}_\alpha(g) = \mathrm{f}^\alpha(g)\mathrm{f}_\alpha(h)$. Since $\mathscr{R}^{-1} = \mathscr{F}^2$ it then immediately follows that

$$h \star g = \overline{R}^\alpha(g) \star \overline{R}_\alpha(h) \ . \tag{8.15}$$

Note 8.1 Since elements of the tensor product $Ug \otimes Ug$ by definition are finite linear combinations of elements $\xi \otimes \zeta$ where $\xi, \zeta \in Ug$, the twist $\mathscr{F}$ strictly speaking does not belong to $Ug \otimes Ug$, because, due to the exponential, an infinite sum over α is understood in expressions (8.10). As we explain in Appendix 7.9, the correct mathematical definition is to introduce a formal parameter h, then, denoting by $\mathbb{C}[[h]]$ the ring of power series in h with coefficients in $\mathbb{C}$, and by $Ug[[h]]$ the algebra of power series in h with coefficients in Ug, we have the well-defined twist $\mathscr{F} = e^{-\frac{i}{2}h\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes_{\mathbb{C}[[h]]}\frac{\partial}{\partial x^\nu}} \in Ug[[h]] \otimes_{\mathbb{C}[[h]]} Ug[[h]]$.

The need for the formal parameter h can also be seen from the definition of the $\star$-product. If f and g are polynomial functions in the x^μ coordinates then $f \star g$ (without formal parameter h) is again a well-defined polynomial function. However, more in general, for smooth functions f and g, the existence of the function $f \star g$ (with no formal parameter h in the $\star$-product) depends on the convergence of the series $f \star g$. Therefore, in this case the $\star$-product is not a well-defined product on the algebra of smooth functions. On the other hand if we work in the deformation quantization context, f and g belong to $\mathrm{Fun}(\mathbb{R}^4)[[h]]$, the algebra of formal power series in h with coefficients in the space of smooth functions $\mathrm{Fun}(\mathbb{R}^4)$. Then $f \star g$ is automatically in $\mathrm{Fun}(\mathbb{R}^4)[[h]]$.

In this book we usually omit writing explicitly the deformation parameter h and we include it in the definition of $\theta^{\mu\nu}$. Moreover frequently we expand the action functional or the equations of motion at a given order n in h (or $\theta^{\mu\nu}$) thus the twists $\mathscr{F}$ and the star product can be approximated by considering only the first n order terms in their $\theta^{\mu\nu}$ expansion.

Note 8.2 We can consider twists and $\star$-products on arbitrary manifolds not just on $\mathbb{R}^4$. For example, given a set of mutually commuting vector fields $\{X_a\}$ ($a = 1, 2, \ldots, n$) on a d-dimensional manifold M, we can consider the twist

$$\mathscr{F} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{ab}X_a \otimes X_b} \ . \tag{8.16}$$

The proof (8.9) of the cocycle condition for $\mathscr{F}$ holds also in this case, indeed for the proof one only needs the Lie algebra elements X_a to be mutually commuting. The class of $\star$-products that can be obtained from these type of twist $\mathscr{F}$ (named abelian twists) is quite rich. For example, on $\mathbb{R}^2$ we can obtain star products that give the commutation relations $x\star y = qy\star x$ with $q \in \mathbb{C}$, and similarly in $\mathbb{R}^n$ (see for example [16]).

Another example of twist is $\mathscr{F} = \mathrm{e}^{\frac{1}{2}H\otimes\ln(1+\lambda E)}$ where the vector fields H and E satisfy $[H,E] = 2E$. In these cases too the $\star$-product defined via (8.5) is associative and properly normalized.

In general an element $\mathscr{F}$ of $Ug\otimes Ug$ is by definition a twist if it is invertible, if it satisfies the cocycle condition (8.3), and if it is properly normalized, i.e.,

$$(id\otimes\varepsilon)\mathscr{F} = (\varepsilon\otimes id)\mathscr{F} = 1\otimes 1 \;, \tag{8.17}$$

where $\varepsilon : Ug \to \mathbb{C}$ is the counit map. ε is the linear and multiplicative map defined by $\varepsilon(1) = 1$ and by $\varepsilon(t) = 0$ for all $t \in g$, cf. (7.34). The normalization condition (8.17) implies the normalization property of the $\star$-product $f\star 1 = 1\star f = f$. On the other hand the cocycle condition (8.3) implies associativity of the $\star$-product.

In the remaining of this note we present a proof of this statement. It can be omitted in a first reading. We begin by inverting relation (8.3) and we obtain $((\Delta\otimes id)\mathscr{F}^{-1})\mathscr{F}_{12}^{-1} = ((id\otimes\Delta)\mathscr{F}^{-1})\mathscr{F}_{23}^{-1}$. Equivalently, using the $\mathscr{F}^{-1} = \bar{\mathrm{f}}^{\alpha}\otimes\bar{\mathrm{f}}_{\alpha}$ notation,

$$\bar{\mathrm{f}}_1^{\alpha}\bar{\mathrm{f}}^{\beta}\otimes\bar{\mathrm{f}}_2^{\alpha}\bar{\mathrm{f}}_{\beta}\otimes\bar{\mathrm{f}}_{\alpha} = \bar{\mathrm{f}}^{\alpha}\otimes\bar{\mathrm{f}}_{\alpha_1}\bar{\mathrm{f}}^{\beta}\otimes\bar{\mathrm{f}}_{\alpha_2}\bar{\mathrm{f}}_{\beta} \;.$$

Here we used Sweedler's notation for the coproduct, for all $\xi \in Ug$, $\Delta(\xi) = \xi_1\otimes\xi_2$ (a sum over ξ_1 and ξ_2 is understood). We recall that the coproduct $\Delta(\xi)$ follows from the coproduct $\Delta(t) = t\otimes 1 + 1\otimes t$ for the Lie algebra elements $t \in g \subset Ug$. This latter coproduct implies the Leibniz rule $t(fh) = t(f)h + ft(h)$, and henceforth $\xi(fh) = \xi_1(f)\xi_2(g)$. Then we compute, for arbitrary functions f,g,h,

$$\begin{aligned}
(f\star g)\star h &= \bar{\mathrm{f}}^{\alpha}(\bar{\mathrm{f}}^{\beta}(f)\bar{\mathrm{f}}_{\beta}(g))\bar{\mathrm{f}}_{\alpha}(h) = (\bar{\mathrm{f}}_1^{\alpha}\bar{\mathrm{f}}^{\beta})(f)(\bar{\mathrm{f}}_2^{\alpha}\bar{\mathrm{f}}_{\beta})(g)\bar{\mathrm{f}}_{\alpha}(h) \\
&= \bar{\mathrm{f}}^{\alpha}(f)(\bar{\mathrm{f}}_{\alpha_1}\bar{\mathrm{f}}^{\beta})(g)(\bar{\mathrm{f}}_{\alpha_2}\bar{\mathrm{f}}_{\beta})(h) = \bar{\mathrm{f}}^{\alpha}(f)\bar{\mathrm{f}}_{\alpha}(\bar{\mathrm{f}}^{\beta}(g)\bar{\mathrm{f}}_{\beta}(h)) \\
&= f\star(g\star h) \;.
\end{aligned}$$

The point of this proof is that it uses only the twist cocycle property (8.3) and property II) stated at the very beginning of this section, i.e., an action of the Lie algebra g (and henceforth of Ug) on the algebra of functions. The algebra of functions is said to be a Ug-module algebra [29] (see also [46], and [16]).

8.2.2 $\star$-Tensor algebra

We now use the twist to deform the commutative geometry on spacetime (vector fields, 1-forms, exterior algebra, tensor algebra, symmetry algebras, covariant derivatives, etc.) into the twisted noncommutative one. The guiding principle is the observation that every time we have a bilinear map

$$\mu : X \times Y \to Z,$$

where X, Y, Z are vector spaces, and where there is an action of the Lie algebra g (and therefore of $\mathscr{F}^{-1}$) on X and Y we can combine this map with the action of the twist. In this way we obtain a deformed version $\mu_\star$ of the initial bilinear map μ:

$$\mu_\star := \mu \circ \mathscr{F}^{-1} , \tag{8.18}$$

$$\begin{aligned} \mu_\star : X \times Y &\to Z \\ (\mathsf{x},\mathsf{y}) &\mapsto \mu_\star(\mathsf{x},\mathsf{y}) = \mu(\bar{\mathrm{f}}^\alpha(\mathsf{x}), \bar{\mathrm{f}}_\alpha(\mathsf{y})) . \end{aligned}$$

The cocycle condition (8.3) implies that if μ is an associative product then also $\mu_\star$ is an associative product.

Algebra of Functions $A_\star$. If $X = Y = Z = Fun(M)$ where $A \equiv Fun(M)$ is the space of functions on spacetime M, we obtain the star product formulae (8.5) and (8.12),

$$f \star g = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(g) = \sum_n \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1} \ldots \theta^{\mu_n\nu_n} \partial_{\mu_1} \ldots \partial_{\mu_n}(f)\, \partial_{\nu_1} \ldots \partial_{\nu_n}(g) . \tag{8.19}$$

The $\star$-product is associative because of the cocycle condition (8.3). We denote by $A_\star$ the noncommutative algebra of functions with the $\star$-product. Notice that to define the $\star$-product we need condition II), the action of the Lie algebra on functions. In this case it is given by the Lie derivative. In the sequel we will always use the Lie derivative action.

Vector fields $\Xi_\star$. We now deform the product $\mu : A \otimes \Xi \to \Xi$ between the space $A = Fun(M)$ of smooth functions on spacetime M and vector fields. A generic vector field is $v = v^\nu \partial_\nu$. Partial derivatives act on vector fields via the Lie derivative action

$$\partial_\mu(v) = [\partial_\mu, v] = \partial_\mu(v^\nu)\partial_\nu . \tag{8.20}$$

According to (8.18) the product $\mu : A \otimes \Xi \to \Xi$ is deformed into the product

$$h \star v = \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(v) . \tag{8.21}$$

Since $\mathscr{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$, iterated use of (8.20) gives

$$h \star v = \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(v) = \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(v^\nu)\partial_\nu = (h \star v^\nu)\partial_\nu . \tag{8.22}$$

It is then easy to see that $h \star (g \star v) = (h \star g) \star v$. We have thus constructed the $A_\star$ module of vector fields. We denote it by $\Xi_\star$. As vector spaces $\Xi = \Xi_\star$, but Ξ is an A module while $\Xi_\star$ is an $A_\star$ module.

1-forms $\Omega_\star$. The space of 1-forms Ω becomes also an $A_\star$ module, with the product between functions and 1-forms given again by following the general prescription (8.18):

$$h \star \omega := \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(\omega) \ . \tag{8.23}$$

The action of $\bar{\mathrm{f}}_\alpha$ on forms is given by iterating the Lie derivative action of the vector field ∂_μ on forms. Explicitly, if $\omega = \omega_\nu dx^\nu$ we have

$$\partial_\mu(\omega) = \partial_\mu(\omega_\nu)dx^\nu \tag{8.24}$$

and $\omega = \omega_\nu dx^\nu = \omega_\mu \star dx^\mu$.

Functions can be multiplied from the left or from the right, if we deform the multiplication from the right we obtain the new product

$$\omega \star h := \bar{\mathrm{f}}^\alpha(\omega)\bar{\mathrm{f}}_\alpha(h) \tag{8.25}$$

and we "move h to the right" with the help of the R-matrix,

$$\omega \star h = \bar{R}^\alpha(h) \star \bar{R}_\alpha(\omega) \ . \tag{8.26}$$

We have defined the $A_\star$-bimodule of 1-forms.

Tensor fields $\mathscr{T}_\star$. Tensor fields form an algebra with the tensor product $\otimes$. We define $\mathscr{T}_\star$ to be the noncommutative algebra of tensor fields. As vector spaces $\mathscr{T} = \mathscr{T}_\star$ the noncommutative tensor product is obtained by applying (8.18):

$$\tau \otimes_\star \tau' := \bar{\mathrm{f}}^\alpha(\tau) \otimes \bar{\mathrm{f}}_\alpha(\tau') \ . \tag{8.27}$$

Associativity of this product follows from the cocycle condition (8.3).

Notice that $\partial_\rho \otimes_\star \partial_\eta = \partial_\rho \otimes \partial_\eta$ because the partial derivatives of $\bar{\mathrm{f}}^\alpha$ applied on ∂_ρ give zero. More in general, if we consider the local coordinate expression of two tensor fields, for example, of the type

$$\tau = \tau^{\mu_1,\dots\mu_m}\partial_{\mu_1} \otimes_\star \dots \partial_{\mu_m},$$

$$\tau' = \tau'^{\nu_1,\dots\nu_n}\partial_{\nu_1} \otimes_\star \dots \partial_{\nu_n},$$

then their $\star$-tensor product is

$$\tau \otimes_\star \tau' = \tau^{\mu_1,\dots\mu_m} \star \tau'^{\nu_1,\dots\nu_n}\partial_{\mu_1} \otimes_\star \dots \partial_{\mu_m} \otimes_\star \partial_{\nu_1} \otimes_\star \dots \partial_{\nu_n} \ . \tag{8.28}$$

There is a natural action of the permutation group on undeformed tensor fields:

$$\tau \otimes \tau' \xrightarrow{\sigma} \tau' \otimes \tau \ .$$

In the deformed case it is the R-matrix that provides a representation of the permutation group on $\star$-tensor fields:

$$\tau \otimes_\star \tau' \xrightarrow{\sigma_{\mathscr{R}}} \overline{R}^\alpha(\tau') \otimes_\star \overline{R}_\alpha(\tau) .$$

It is easy to check that, consistently with $\sigma_{\mathscr{R}}$ being a representation of the permutation group, we have $(\sigma_{\mathscr{R}})^2 = id$. Indeed we have $\mathscr{R}^{-1} = \mathscr{R}_{21}$, i.e., the $\mathscr{R}$-matrix is triangular.

Exterior forms $\boldsymbol{\Omega^{\cdot}_\star = \oplus_p \Omega^p_\star}$. Exterior forms form an algebra with product $\wedge : \Omega^{\cdot} \times \Omega^{\cdot} \to \Omega^{\cdot}$. We $\star$-deform the wedge product into the $\star$-wedge product,

$$\vartheta \wedge_\star \vartheta' := \overline{\mathrm{f}}^\alpha(\vartheta) \wedge \overline{\mathrm{f}}_\alpha(\vartheta') . \tag{8.29}$$

We denote by $\Omega^{\cdot}_\star$ the linear space of forms equipped with the wedge product $\wedge_\star$.

As in the commutative case exterior forms are totally $\star$-antisymmetric contravariant tensor fields. For example, the 2-form $\omega \wedge_\star \omega'$ is the $\star$-antisymmetric combination

$$\omega \wedge_\star \omega' = \omega \otimes_\star \omega' - \overline{R}^\alpha(\omega') \otimes_\star \overline{R}_\alpha(\omega) . \tag{8.30}$$

Since the Lie derivative and the exterior derivative commute, the usual exterior derivative $d : A \to \Omega$ satisfies the Leibniz rule

$$d(h \star g) = dh \star g + h \star dg \tag{8.31}$$

and is therefore also the $\star$-exterior derivative. On higher forms too the usual exterior derivative satisfies the Leibniz rule $d(\vartheta \wedge_\star \vartheta') = d\vartheta \wedge_\star \vartheta' + (-1)^{|\vartheta|} \vartheta \wedge_\star d\vartheta'$ and is therefore also the $\star$-exterior derivative. Due to the commutativity between Lie derivative and exterior derivative it turns out that the de Rham cohomology ring is undeformed.

$\star$-Pairing between 1-forms and vector fields. We now consider the bilinear map

$$\langle\ ,\ \rangle : \Xi \times \Omega \to A , \tag{8.32}$$

$$(v, \omega) \ \mapsto \langle v, \omega \rangle = \langle v^\mu \partial_\mu, \omega_\nu dx^\nu \rangle = v^\mu \omega_\mu . \tag{8.33}$$

Always according to the general prescription (8.18) we deform this pairing into

$$\langle\ ,\ \rangle_\star : \Xi_\star \times \Omega_\star \to A_\star , \tag{8.34}$$

$$(\xi, \omega) \ \mapsto \langle \xi, \omega \rangle_\star := \langle \overline{\mathrm{f}}^\alpha(\xi), \overline{\mathrm{f}}_\alpha(\omega) \rangle . \tag{8.35}$$

It is easy to see that the $\star$-pairing satisfies the $A_\star$-linearity properties

$$\langle h \star u, \omega \star k \rangle_\star = h \star \langle u, \omega \rangle_\star \star k , \tag{8.36}$$

$$\langle u, h \star \omega \rangle_\star = \overline{R}^\alpha(h) \star \langle \overline{R}_\alpha(u), \omega \rangle_\star . \tag{8.37}$$

Notice that $\langle \partial_\mu, dx^\nu \rangle_\star = \langle \partial_\mu, dx^\nu \rangle = \delta^\nu_\mu$.

Using the pairing $\langle\ ,\ \rangle_\star$ we associate to any 1-form ω the left $A_\star$-linear map $\langle\ ,\omega\rangle_\star$. Also the converse holds: any left $A_\star$-linear map $\Phi : \Xi_\star \to A_\star$ is of the form $\langle\ ,\omega\rangle_\star$ for some ω (explicitly $\omega = \Phi(\partial_\mu)dx^\mu$).

Note 8.3 In order to understand the coordinate independence of expression (8.19) it is helpful to rewrite it using the notation (8.7),

$$f \star g = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(g) = \sum_n \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{a_1b_1}\ldots\theta^{a_nb_n}X_{a_1}\ldots X_{a_n}(f)\,X_{b_1}\ldots X_{b_n}(g)\ . \tag{8.38}$$

Let us study the first order in θ term, $\frac{i}{2}\theta^{ab}X_a(f)X_b(g)$. The expression $X_a(f)$ denotes the Lie derivative of the global vector field X_a on the function f (globally defined on the spacetime manifold) and therefore $X_a(f)$ is a new globally defined function; similarly $X_b(g)$. Then also $\frac{i}{2}\theta^{ab}X_a(f)X_b(g)$ is a new globally defined function because it is a linear combination, with constant coefficients θ^{ab} of globally defined functions. Similarly also $X_{a_1}(\ldots X_{a_n}(f)\ldots)$ is a globally defined (coordinate-independent) function on spacetime. In the $\{x^\mu\}$ reference frame it simply reads $\frac{\partial}{\partial x^{\mu_1}}(\ldots\frac{\partial}{\partial x^{\mu_n}}(f)\ldots)$; in another coordinate system $\{y^\nu\}$ it reads $e_{\mu_1}^{\nu_1}\frac{\partial}{\partial y^{\mu_1}}(\ldots e_{\nu_n}^{\nu_n}\frac{\partial}{\partial y^{\nu_n}}(f)\ldots)$, where $\frac{\partial}{\partial x^\mu} = e_\mu^\nu(y)\frac{\partial}{\partial y^\nu}$. The transformation $x^\mu \to y^\mu$ is a finite coordinate transformation on commutative spacetime. Later on we study infinitesimal *noncommutative* coordinate transformations.

The transformation properties of expression (8.21) are shown by using arguments similar to those after (8.38). The important point is that according to the notation (8.7), the partial derivatives present in the twist are a specific choice of (globally defined) vector fields X_a. These vector fields act on (globally defined) vector fields v via the Lie derivative action, $X_a(v) = [X_a, v]$, the formalism is geometric, $[X_a, v]$ is a new (globally defined) vector field.

8.2.3 $\star$-Diffeomorphism symmetry

The twist deformation program of the previous section can be further developed and we can study the deformed symmetry transformations acting on deformed tensor fields. The appropriate language for the study of symmetries in this context is that of Hopf algebras.

$\star$-Hopf algebra of diffeomorphisms $U\Xi_\star$. We recall that the (infinite-dimensional) linear space Ξ of smooth vector fields on spacetime M becomes a Lie algebra through the map

$$\begin{aligned} [\ \ ] : \quad & \Xi \times \Xi \to \Xi \\ & (u, v) \mapsto [u\ v]. \end{aligned} \tag{8.39}$$

The element $[u\ v]$ of Ξ is defined by the usual Lie bracket

$$[u\ v](h) = u(v(h)) - v(u(h)), \tag{8.40}$$

where h is a function on spacetime.

The Lie algebra of vector fields (i.e., the algebra of infinitesimal local diffeomorphisms) can also be seen as an abstract Lie algebra without referring to the action of vector fields on functions. The universal enveloping algebra $U\Xi$ of this abstract Lie algebra is the associative algebra (over $\mathbb{C}$) generated by the elements of Ξ and the unit element 1 and where the element $[u\ v]$ is given by the commutator $uv - vu$, i.e., $uv - vu = [u\ v]$. Here uv and vu denote the product in $U\Xi$. The algebra $U\Xi$ is the universal enveloping algebra of vector fields (infinitesimal local diffeomorphisms), we shall denote its generic elements (sums of products of vector fields $u \in \Xi$) by the letters ξ, ζ, η,....

The undeformed algebra $U\Xi$ has a natural Hopf algebra structure [44–46]. On the generators $u \in \Xi$ the coproduct map Δ, the counit ε and the antipode S are defined by

$$\begin{aligned} \Delta(u) &= u \otimes 1 + 1 \otimes u\ , \\ \varepsilon(u) &= 0 \\ S(u) &= -u \end{aligned} \tag{8.41}$$

(and $\Delta(1) = 1 \otimes 1,\ \varepsilon(1) = 1,\ S(1) = 1$). The maps Δ and ε are then extended as algebra homomorphisms and S as antialgebra homomorphism to the full enveloping algebra, $\Delta : U\Xi \to U\Xi \otimes U\Xi$, $\varepsilon : U\Xi \to \mathbb{C}$, and $S : U\Xi \to U\Xi$,

$$\begin{aligned} \Delta(\xi\zeta) &:= \Delta(\xi)\Delta(\zeta)\ , \\ \varepsilon(\xi\zeta) &:= \varepsilon(\xi)\varepsilon(\zeta)\ , \\ S(\xi\zeta) &:= S(\zeta)S(\xi)\ . \end{aligned} \tag{8.42}$$

The extensions of Δ, ε, and S are well defined because they are compatible with the relations $uv - vu = [u\ v]$ (for example, $S(uv - vu) = S(v)S(u) - S(u)S(v) = -[u\ v] = S[u\ v]$).

On the generators, the coproduct encodes the Leibniz rule property $u(hg) = u(h)g + hu(g)$, the antipode expresses the fact that the inverse of the group element e^u is e^{-u}, while the counit associates to every element e^u the identity 1.

In order to construct the deformed algebra of diffeomorphisms we apply the recipe (8.18) and deform the product in $U\Xi$ into the new product

$$\xi \star \zeta = \bar{\mathrm{f}}^{\alpha}(\xi)\bar{\mathrm{f}}_{\alpha}(\zeta)\ . \tag{8.43}$$

We call $U\Xi_\star$ the new algebra with product $\star$, as vector spaces $U\Xi = U\Xi_\star$. Since any sum of products of vector fields in $U\Xi$ can be rewritten as sum of $\star$-products of vector fields via the formula $uv = \mathrm{f}^{\alpha}(u) \star \mathrm{f}_{\alpha}(v)$, vector fields u generate the algebra $U\Xi_\star$.

It turns out [16] that $U\Xi_\star$ has also a natural Hopf algebra structure. We describe it by giving the coproduct, the counit, and the antipode[2] on the generators u of $U\Xi_\star$:

$$\Delta_\star(u) = u \otimes 1 + \overline{R}^\alpha \otimes \overline{R}_\alpha(u) \;, \tag{8.44}$$

$$\varepsilon_\star(u) = \varepsilon(u) = 0 \;, \tag{8.45}$$

$$S_\star(u) = -\overline{R}^\alpha(u)\overline{R}_\alpha \;. \tag{8.46}$$

In Appendix 8.7 we prove for example coassociativity of the coproduct $\Delta_\star$. We here show that the coproduct definition (8.44) can be inferred from a deformed Leibniz rule.

There is a natural action (Lie derivative) of $\Xi_\star$ on the space of functions $A_\star$. It is given once again by combining the usual Lie derivative on functions $\mathscr{L}_u(h) = u(h)$ with the twist $\mathscr{F}$ as in (8.18),

$$\mathscr{L}^\star_u(h) := \overline{\mathrm{f}}^\alpha(u)(\overline{\mathrm{f}}_\alpha(h)) \;. \tag{8.47}$$

By recalling that every vector field can be written as $u = u^\mu \star \partial_\mu = u^\mu \partial_\mu$ we have

$$\begin{aligned} \mathscr{L}^\star_u(h) &= \overline{\mathrm{f}}^\alpha(u^\mu \partial_\mu)(\overline{\mathrm{f}}_\alpha(h)) = \overline{\mathrm{f}}^\alpha(u^\mu)\,\partial_\mu(\overline{\mathrm{f}}_\alpha(h)) \\ &= u^\mu \star \partial_\mu(h) \;, \end{aligned} \tag{8.48}$$

where in the second equality we have considered the explicit expression (8.11) of $\overline{\mathrm{f}}^\alpha$ in terms of partial derivatives, and we have iteratively used the property $[\partial_\nu, u^\mu \partial_\mu] = \partial_\nu(u)\,\partial_\mu$. In the last equality we have used that the partial derivatives contained in $\overline{\mathrm{f}}_\alpha$ commute with the partial derivative ∂_μ.

In accordance with the coproduct formula (8.44) the differential operator $\mathscr{L}^\star_u$ satisfies the deformed Leibniz rule

$$\mathscr{L}^\star_u(h \star g) = \mathscr{L}^\star_u(h) \star g + \overline{R}^\alpha(h) \star \mathscr{L}^\star_{\overline{R}_\alpha(u)}(g) \;. \tag{8.49}$$

Indeed recalling that $u = u^\mu \star \partial_\mu = u^\mu \partial_\mu$ we have

$$\begin{aligned} \mathscr{L}^\star_u(h \star g) = u^\mu \star \partial_\mu(h \star g) &= u^\mu \star \partial_\mu(h) \star g + u^\mu \star h \star \partial_\mu(g) \\ &= \mathscr{L}^\star_u(h) \star g + \overline{R}^\alpha(h) \star \overline{R}_\alpha(u^\mu) \star \partial_\mu(g) \\ &= \mathscr{L}^\star_u(h) \star g + \overline{R}^\alpha(h) \star \mathscr{L}^\star_{\overline{R}_\alpha(u)}(g) \;. \end{aligned} \tag{8.50}$$

From (8.48) it is also immediate to check the compatibility condition

$$\mathscr{L}^\star_{f \star u}(h) = f \star \mathscr{L}^\star_u(h) \tag{8.51}$$

[2] Notice that because of the antisymmetry of $\theta^{\mu\nu}$ we have $\overline{R}^\alpha(u)\overline{R}_\alpha = \overline{R}_\alpha\ \overline{R}^\alpha(u)$. Since $S_\star(\partial_\nu) = -\partial_\nu$ it is then easy to prove that $S_\star^2 = id$. It is also easy to check that $\mu(S \otimes id)\Delta(u) = \mu(id \otimes S)\Delta(u) = \varepsilon(u)1 = 0$. This last property uniquely defines the antipode.

that shows that the action $\mathscr{L}^\star$ is the one compatible with the $A_\star$ module structure of vector fields.

The action $\mathscr{L}^\star$ of $\Xi_\star$ on $A_\star$ can be extended to all $U\Xi_\star$. We recall that the action of $U\Xi$ on the space of functions can be defined by extending the Lie derivative. For any function $h \in A = Fun(M)$, we define the Lie derivative of a product of generators $u...vz$ in $U\Xi$ to be the composition of the Lie derivatives of the generators,

$$(u...vz)(h) = u(\ldots v(z(h))...) \ . \tag{8.52}$$

Then by linearity we know the Lie derivative along any element ξ of $U\Xi$. We then define

$$\mathscr{L}^\star_\xi(h) := \bar{\mathrm{f}}^\alpha(\xi)(\bar{\mathrm{f}}_\alpha(h)) \ . \tag{8.53}$$

The map $\mathscr{L}^\star$ is an action of $U\Xi_\star$ on $A_\star$, i.e., it represents the algebra $U\Xi_\star$ as differential operators on functions because

$$\mathscr{L}^\star_u(\mathscr{L}^\star_v(h)) = \mathscr{L}^\star_{u\star v}(h) \ . \tag{8.54}$$

$\star$-Lie algebra of vector fields $\Xi_\star$. We now turn our attention to the issue of determining the Lie algebra $\Xi_\star$ of $U\Xi_\star$. In the undeformed case the Lie algebra of the universal enveloping algebra $U\Xi$ is the linear subspace Ξ of $U\Xi$ of primitive elements, i.e., of elements u that have coproduct:

$$\Delta(u) = u \otimes 1 + 1 \otimes u \ . \tag{8.55}$$

Of course Ξ generates $U\Xi$ and Ξ is closed under the usual commutator bracket $[\ ,\ ]$,

$$[u,v] = uu - vu \in \Xi \quad \text{for all } u,v \in \Xi \ . \tag{8.56}$$

The geometric meaning of the bracket $[u,v]$ is that it is the adjoint action of Ξ on Ξ,

$$[u,v] = ad_u\, v, \tag{8.57}$$

$$ad_u\, v := u_1 v S(u_2), \tag{8.58}$$

where we have used the notation $\Delta(u) = u_1 \otimes u_2$, where a sum over u_1 and u_2 is understood. Recalling that $\Delta(u) = u \otimes 1 + 1 \otimes u$ and that $S(u) = -u$, from (8.58) we immediately obtain (8.57). In other words, the commutator $[u,v]$ is the Lie derivative of the left invariant vector field u on the left invariant vector field v. More in general the adjoint action of $U\Xi$ on $U\Xi$ is given by

$$ad_\xi\, \zeta = \xi_1 \zeta S(\xi_2) \ , \tag{8.59}$$

where we used the notation (sum understood)

$$\Delta(\xi) = \xi_1 \otimes \xi_2 \ .$$

For example, $ad_{uv}\, \zeta = [u,[v,\zeta]]$.

In the deformed case the coproduct is no more cocommutative and we cannot identify the Lie algebra of $U\Xi_\star$ with the primitive elements of $U\Xi$, they are too few.[3] There are three natural conditions that according to [47] the $\star$-Lie algebra of $U\Xi_\star$ has to satisfy (see Chap. 7). It has to be a linear subspace $\Xi_\star$ of $U\Xi_\star$ such that

$$i)\ \ \Xi_\star \text{ generates } U\Xi_\star\ , \tag{8.60}$$

$$ii)\ \ \Delta_\star(\Xi_\star) \subset \Xi_\star \otimes 1 + U\Xi_\star \otimes \Xi_\star\ , \tag{8.61}$$

$$iii)\ \ [\Xi_\star, \Xi_\star]_\star \subset \Xi_\star\ . \tag{8.62}$$

Property $ii)$ implies a minimal deformation of the Leibniz rule. Property $iii)$ is the closure of $\Xi_\star$ under the adjoint action:

$$[u,v]_\star = ad^\star_u\, v = u_{1_\star} \star v \star S(u_{2_\star})\ , \tag{8.63}$$

here we have used the coproduct notation $\Delta_\star(u) = u_{1_\star} \otimes u_{2_\star}$. More in general the adjoint action is given by

$$ad^\star_\xi\, \zeta := \xi_{1_\star} \star \zeta \star S_\star(\xi_{2_\star})\ , \tag{8.64}$$

where we used the coproduct notation $\Delta_\star(\xi) = \xi_{1_\star} \otimes \xi_{2_\star}$.

In the case the deformation is given by a twist we have a natural candidate for the Lie algebra of the Hopf algebra $U\Xi_\star$. We apply the recipe (8.18) and deform the Lie algebra product $[\ \ ]$ given in (8.39) into

$$\begin{aligned} [\ \ ]_\star : \qquad & \Xi \times \Xi \to \Xi \\ & (u,v) \mapsto [u\ v]_\star := [\bar{\mathrm{f}}^\alpha(u)\ \bar{\mathrm{f}}_\alpha(v)]\ . \end{aligned} \tag{8.65}$$

In $U\Xi_\star$ this $\star$-Lie bracket can be realized as a deformed commutator

$$\begin{aligned} [u\ v]_\star &= [\bar{\mathrm{f}}^\alpha(u)\ \bar{\mathrm{f}}_\alpha(v)] = \bar{\mathrm{f}}^\alpha(u)\bar{\mathrm{f}}_\alpha(v) - \bar{\mathrm{f}}_\alpha(v)\bar{\mathrm{f}}^\alpha(u) \\ &= u \star v - \bar{R}^\alpha(v) \star \bar{R}_\alpha(u)\ . \end{aligned} \tag{8.66}$$

It is easy to see that the bracket $[\ \ ]_\star$ has the $\star$-antisymmetry property

$$[u\ v]_\star = -[\bar{R}^\alpha(v)\ \bar{R}_\alpha(u)]_\star\ . \tag{8.67}$$

This can be shown as follows:

$$[u\ v]_\star = [\bar{\mathrm{f}}^\alpha(u)\ \bar{\mathrm{f}}_\alpha(v)] = -[\bar{\mathrm{f}}_\alpha(v)\ \bar{\mathrm{f}}^\alpha(u)] = -[\bar{R}^\alpha(v)\ \bar{R}_\alpha(u)]_\star\ .$$

[3] This can already be seen at the semiclassical level, where we are left with the symplectic structure. Primitive elements then correspond to symplectic infinitesimal transformations. Instead of restricting the set of transformations to those compatible with the bivector $\theta^{\mu\nu}$ we want to properly generalize/relax the notion of infinitesimal automorphism. In this way we do not consider $\theta^{\mu\nu}$ as the components of a bivector, but as a set of constant coefficients.

A $\star$-Jacobi identity can be proven as well

$$[u\,[v\,z]_\star]_\star = [[u\,v]_\star\,z]_\star + [\overline{R}^\alpha(v)\,[\overline{R}_\alpha(u)\,z]_\star]_\star\,. \tag{8.68}$$

The appearance of the R-matrix $\mathscr{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha$ is not unexpected. We have seen that $\mathscr{R}^{-1}$ encodes the noncommutativity of the $\star$-product $h\star g = \overline{R}^\alpha(g)\star\overline{R}_\alpha(h)$ so that $h\star g$ do $\mathscr{R}^{-1}$-commute. Then it is natural to define $\star$-commutators using the $\mathscr{R}^{-1}$ matrix. In other words, the representation of the permutation group to be used on twisted noncommutative spaces is the one given by the $\mathscr{R}^{-1}$ matrix.

We now show that the subspace $\Xi_\star$ (that as vector space equals Ξ) has all the three properties i), ii), iii). It satisfies i) because any sum of products of vector fields in $U\Xi$ can be rewritten as sum of $\star$-products of vector fields via the formula $uv = \mathrm{f}^\alpha(u)\star\mathrm{f}_\alpha(v)$, and therefore $\star$-vector fields generate the algebra. It obviously satisfies ii), and finally in Appendix 8.8 we prove that it satisfies iii) by showing that the bracket $[u\,v]_\star$ is indeed the adjoint action, $ad^\star_u v = [u\,v]_\star$.

We stress that the geometrical – and therefore physical – interpretation of $\Xi_\star$ as infinitesimal diffeomorphisms is due to the deformed Leibniz rule property ii) and to the closure of $\Xi_\star$ under the adjoint action. Property ii) will be fundamental in order to define covariant derivatives (cf. (8.102)).

Note 8.4 The Hopf algebra $U\Xi_\star$ can be described via the generators $X_u := \mathrm{f}^\alpha(u)\mathrm{f}_\alpha$ rather than via the u generators. The action of X_u on functions is the differential operator $X^\star_u \equiv \mathscr{L}^\star_{X_u}$, we have $X^\star_u(f) \equiv \mathscr{L}^\star_{X_u}(f) = u(f)$, compare with Chap. 1, Sect. 1.5 and Chap. 3, Sect. 3.2, see also Eq. (5.2) in [15]. The generators X_u satisfy the commutation relations $X_u\star X_v - X_v\star X_u = X_{[u,v]}$ and their coproduct is $\Delta_\star(X_u) = \mathscr{F}(X_u\otimes 1 + 1\otimes X_u)\mathscr{F}^{-1}$. We see that $U\Xi_\star$ is the abstract Hopf algebra of diffeomorphisms considered in [15], end of Sect. 5. Since the elements X_u generate $U\Xi_\star$, invariance under the diffeomorphisms algebra $U\Xi_\star$ is equivalently shown by proving invariance under the X_u or the u generators. Since $X_{\partial_\mu} = \partial_\mu$ partial derivatives belong to both sets of generators. We also have $\mathscr{L}^\star_{\partial_\mu}(f) = \partial_\mu(f) = \partial^\star_\mu f$.

$\star$-Infinitesimal transformations. In the commutative case the diffeomorphisms algebra $U\Xi$ acts on the algebra of functions and more in general on the algebra of tensor fields via the Lie derivative. The Riemann curvature, the Ricci tensor, and the curvature scalar are tensors and therefore they transform covariantly under the diffeomorphisms action. In the twisted case, the $\star$-diffeomorphisms algebra $U\Xi_\star$ acts on the $\star$-algebra of functions $A_\star$ and more in general on the $\star$-algebra of tensor fields $\mathscr{T}_\star$. The action on functions is given by the $\star$-Lie derivative defined in (8.47). Similarly the action on tensors is given, according to (8.18), by

$$\mathscr{L}^\star_u(\tau) := \overline{\mathrm{f}}^\alpha(u)(\overline{\mathrm{f}}_\alpha(\tau))\,. \tag{8.69}$$

This expression defines an action because $\mathscr{L}^\star_u(\mathscr{L}^\star_v(\tau)) = \mathscr{L}^\star_{u\star v}(\tau)$. In particular the $\star$-Lie derivative is a representation of the $\star$-Lie algebra of infinitesimal diffeomorphisms $\Xi_\star$,

$$\mathscr{L}^\star_u \mathscr{L}^\star_v - \mathscr{L}^\star_{\overline{R}^\alpha(v)} \mathscr{L}^\star_{\overline{R}_\alpha(u)} = \mathscr{L}^\star_{[u\ v]_\star} , \tag{8.70}$$

where $\mathscr{L}^\star_u \mathscr{L}^\star_v = \mathscr{L}^\star_u \circ \mathscr{L}^\star_v$ is the usual composition of operators. The coproduct in $U\Xi$ is compatible with the product in the tensor fields algebra because

$$\mathscr{L}^\star_u(\tau \otimes_\star \tau') = \mathscr{L}^\star_u(\tau) \star \tau' + \overline{R}^\alpha(\tau) \star \mathscr{L}^\star_{\overline{R}_\alpha(u)}(\tau') . \tag{8.71}$$

In Sect. 8.4 we introduce the noncommutative Riemann tensor and Ricci curvature and show that they are indeed tensors. Then they transform covariantly under the action of the $\star$-diffeomorphism algebra. The corresponding noncommutative Einstein equations satisfy the symmetry principle of noncommutative general covariance, i.e., they are covariant under $\star$-diffeomorphism symmetry.

8.2.3.1 Relation between $U\Xi_\star$ and $U\Xi^{\mathscr{F}}$

In the previous four pages, using the twist $\mathscr{F}$ and the general prescription (8.18) we have described the Hopf algebra

$$(U\Xi_\star, \star, \Delta_\star, S_\star, \varepsilon)$$

and its Lie algebra $(\Xi_\star, [\ \ ]_\star)$. These are a deformation of the cocommutative Hopf algebra

$$(U\Xi, \cdot, \Delta, S, \varepsilon)$$

and its Lie algebra $(\Xi, [\ \ ])$. Usually given a twist $\mathscr{F}$ one deforms the Hopf algebra $(U\Xi, \cdot \Delta, S, \varepsilon)$ into the Hopf algebra

$$(U\Xi^{\mathscr{F}}, \cdot, \Delta^{\mathscr{F}}, S^{\mathscr{F}}, \varepsilon)$$

where the coproduct is deformed via

$$\Delta^{\mathscr{F}}(\xi) := \mathscr{F}\Delta(\xi)\mathscr{F}^{-1} , \tag{8.72}$$

while product, antipode, and counit are undeformed $\cdot^{\mathscr{F}} = \cdot\,, S^{\mathscr{F}} = S\,, \varepsilon^{\mathscr{F}} = \varepsilon$ ($S^{\mathscr{F}} = S$ only for abelian antisymmetric twists).

The Hopf algebras $U\Xi_\star$ and $U\Xi^{\mathscr{F}}$ are isomorphic, as vector spaces $U\Xi_\star = U\Xi = U\Xi^{\mathscr{F}}$. The Hopf algebra isomorphism is given by the linear map $D : U\Xi_\star \to U\Xi_\star$

$$D(\xi) = \overline{\mathrm{f}}^\alpha(\xi)\overline{\mathrm{f}}_\alpha . \tag{8.73}$$

The inverse of the map D is

$$D^{-1} \equiv X : \xi \longmapsto X_\xi = \mathrm{f}^\alpha(\xi)\mathrm{f}_\alpha ,$$

indeed $D(X_\xi) = \bar{\mathrm{f}}^\beta(\mathrm{f}^\alpha(\xi)\mathrm{f}_\alpha)\bar{\mathrm{f}}_\beta = \bar{\mathrm{f}}^\beta(\mathrm{f}^\alpha(\xi))\mathrm{f}_\alpha\bar{\mathrm{f}}_\beta = (\bar{\mathrm{f}}^\beta\mathrm{f}^\alpha)(\xi)\mathrm{f}_\alpha\bar{\mathrm{f}}_\beta = \xi$ where we used that partial derivatives commute among themselves and in the last line we used $\mathscr{F}^{-1}\mathscr{F} = 1 \otimes 1$. Explicitly the Hopf algebra isomorphisms between $U\Xi_\star$ and $U\Xi^{\mathscr{F}}$ is [16]

$$D(\xi \star \zeta) = D(\xi)D(\zeta) \; , \tag{8.74}$$

$$\Delta_\star = (D^{-1} \otimes D^{-1}) \circ \Delta^{\mathscr{F}} \circ D \; , \tag{8.75}$$

$$S_\star = D^{-1} \circ S^{\mathscr{F}} \circ D \; . \tag{8.76}$$

Under this isomorphism the Lie algebra $\Xi_\star$ is mapped into the Lie algebra $\Xi^{\mathscr{F}} := D(\Xi_\star)$ of all elements

$$u^{\mathscr{F}} := D(u) = \bar{\mathrm{f}}^\alpha(u)\bar{\mathrm{f}}_\alpha \; .$$

The bracket in $\Xi^{\mathscr{F}}$ is the deformed commutator

$$[u^{\mathscr{F}}, v^{\mathscr{F}}]_{\mathscr{F}} = u^{\mathscr{F}} v^{\mathscr{F}} - \overline{R}^\alpha(v^{\mathscr{F}})\overline{R}_\alpha(u^{\mathscr{F}}) \tag{8.77}$$

and it equals the adjoint action in $U\Xi^{\mathscr{F}}$,

$$[u^{\mathscr{F}}, v^{\mathscr{F}}]_{\mathscr{F}} = ad^{\mathscr{F}}_{u^{\mathscr{F}}} v^{\mathscr{F}} = u^{\mathscr{F}}_{1_{\mathscr{F}}} v S(u^{\mathscr{F}}_{2_{\mathscr{F}}}) \; , \tag{8.78}$$

where we used the notation $\Delta^{\mathscr{F}}(\xi) = \xi_{1_{\mathscr{F}}} \otimes \xi_{2_{\mathscr{F}}}$. The usual Lie algebra Ξ of vector fields with the usual bracket $[u,v] = uv - vu$ is not properly a Lie algebra of $U\Xi^{\mathscr{F}}$ because the commutator fails to be the adjoint action and the Leibniz rule is not of the type *ii*). In particular the vector fields u have not the geometric interpretation of infinitesimal diffeomorphisms.

8.2.4 Twisted versus spontaneously broken symmetries

Given the deformation $A_\star = Fun_\star(M)$ of the algebra of functions $A = Fun(M)$, one can

- consider the derivations of $Fun_\star(M)$, i.e., the infinitesimal transformations of $Fun_\star(M)$ that satisfy the usual Leibniz rule, $u(h \star g) = u(h) \star g + h \star u(g)$. As is easily seen expanding in power series of $\theta^{\mu\nu}$, these maps are only the vector fields that leave invariant the Poisson tensor (8.6). Thus while in the commutative case any vector field is a derivation, in the deformed case the space of derivations is smaller. This is the usual viewpoint, considered for example in the quantization deformation paper [48]. This viewpoint for our purposes is too restrictive, for example, infinitesimal Poincaré transformations are not derivations. In this approach we have that Poincaré invariance is spontaneously broken by the presence of (the background field) $\theta^{\mu\nu}$.

- consistently deform the notion of derivation so that to any infinitesimal transformation of $Fun(M)$ there corresponds one and only one deformed infinitesimal derivation. This is what we have achieved with the map $\mathscr{L}_u \to \mathscr{L}_u^\star$, where $\mathscr{L}_u^\star$ satisfies the deformed Leibniz rule (7.55). This is the quantum groups and quantum spaces approach [9, 12, 18, 49]. The bonus of this approach is that instead of dealing with a spontaneously broken diffeomorphisms (or Poincaré) symmetry we have an unbroken quantum diffeomorphisms (or Poincaré) symmetry. In this way we retain a symmetry property that is as strong as the one of commutative spacetime. This is doable if the $\star$-product deformation of $Fun(M)$ can be obtained from a twist. This is the twisted symmetry approach.

These two approaches coexist and are equally tenable viewpoints in order to understand the infinitesimal transformation of the star product of two functions $f \star g$ (more in general of fields).

Let us consider $\mathscr{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}$ with $a,b = 0,\ldots,3$ and X_a globally defined and mutually commuting vector fields (for example, $X_0 = \partial_0$, $X_1 = \partial_1$, $X_2 = \partial_2$, $X_3 = \partial_3$). The deformed coproduct $\Delta^{\mathscr{F}}(u)$ absorbs exactly the variation $[u, X_a]$ under u of the vector fields X_a present in the star product $f \star g$. In the first approach we say that X_a (or $\theta^{ab}X_a \otimes X_b$) changes under the infinitesimal transformation generated by u. Thus u is the generator of a broken symmetry. In the second approach the change in X_a is reinterpreted as a change in the Leibniz rule for u. Now X_a does not change, therefore u is indeed a symmetry transformation of the Hopf algebra $U\Xi^{\mathscr{F}}$.

The reader can check these two viewpoints with an explicit calculation by expanding in power series of θ, or with the following one

$$\begin{aligned} u(f \star g) &= u(\bar{\mathrm{f}}^\gamma(f)\bar{\mathrm{f}}_\gamma(g)) \\ &= u_1(\bar{\mathrm{f}}^\gamma(f))\, u_2(\bar{\mathrm{f}}_\gamma(g)) \qquad (8.79) \\ &= (\bar{\mathrm{f}}^\alpha \mathrm{f}^\beta u_1 \bar{\mathrm{f}}^\gamma)(f)\, (\bar{\mathrm{f}}_\alpha \mathrm{f}_\beta u_2 \bar{\mathrm{f}}_\gamma)(g) \\ &= u_1^{\mathscr{F}}(f)\, u_2^{\mathscr{F}}(g), \qquad (8.80) \end{aligned}$$

where we used Sweedler's notation (sum over the indices 1 and 2 understood)

$$\Delta(u) = u_1 \otimes u_2 = u \otimes 1 + 1 \otimes u \ , \quad \Delta^{\mathscr{F}}(u) = \mathscr{F}\Delta(u)\mathscr{F}^{-1} = u_1^{\mathscr{F}} \otimes u_2^{\mathscr{F}} \ ,$$

and in the third equality we inserted $1 \otimes 1 = \mathscr{F}^{-1}\mathscr{F} = \bar{\mathrm{f}}^\alpha \mathrm{f}^\beta \otimes \bar{\mathrm{f}}_\alpha \mathrm{f}_\beta$.

In (8.79) the Leibniz rule is undeformed, but u_1 and u_2 act also on $\mathscr{F}^{-1}$. In (8.80) the Leibniz rule is deformed and $u_1^{\mathscr{F}}$, $u_2^{\mathscr{F}}$ act directly on f and g and not on the $\star$-product.

It is using this second approach, and the symmetry described by the Hopf algebra $U\Xi_\star$ that is isomorphic to $U\Xi^{\mathscr{F}}$, that we are able to construct a gravity theory. The point is that this twisted symmetry approach holds only as long as the vector fields X_a enter the formalism just inside the $\star$-product. This is precisely what we want. This approach is therefore more powerful because among the different spontaneously broken theories ($\theta \neq 0$) it singles out those where θ enters only through the star product.

A related issue is that of the construction of conserved currents associated with noncommutative spacetime symmetries. A noncommutative Noether procedure is presently missing and further work is required in this direction. Here we just mention that in the usual undeformed gravity we have the covariant conservation of the Einstein tensor $\mathsf{Ric}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathsf{R}$. This is a consequence of the Bianchi identity for the curvature tensor, an identity that can also be proven in the noncommutative case with arbitrary twist $\mathscr{F}$ [27] (not necessarily a twist which is invariant under spacetime translations).

On the other hand, even if the action functional of scalar field theories on flat noncommutative four-dimensional Minkowski spacetime is invariant under deformed Poincaré transformations (see [15] Sect. 6), accordingly with [50] the presently known way to construct a conserved angular momentum tensor is to enhance the noncommutativity parameter $\theta^{\mu\nu}(x)$ to a dynamical field [51].

The difference between the scalar field theory case and the gravity case may reside in the different ways the two theories are constructed. In the gravity case the noncommutative diffeomorphisms dictate the field equations. In the scalar case the theory is obtained from the commutative one by replacing the usual product of fields with the $\star$-product, this procedure is less geometric and might be responsible for the present absence of the conserved currents associated with deformed Lorentz rotations. A first step toward a deeper understanding of the open issue of the Noether theorem in noncommutative spacetimes is that of gaining a good command of the notion of infinitesimal symmetry transformation and in particular of infinitesimal Poincaré transformation. As we show in the next section this is a well-understood notion.

8.3 Poincaré symmetry

The considerations about the undeformed Hopf algebra $U\Xi$ and the Hopf algebras $U\Xi_\star$ and $U\Xi^{\mathscr{F}}$ hold independently from Ξ being the Lie algebra of infinitesimal diffeomorphisms. In this section we study the case of the deformed Poincaré algebra. It can be seen as an abstract algebra or also as a subalgebra of infinitesimal diffeomorphisms Ξ.

8.3.1 $\star$-Poincaré algebra

We start by recalling that the usual Poincaré Lie algebra $iso(3,1)$:

$$[P_\mu, P_\nu] = 0 \,,$$

$$[P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \,, \tag{8.81}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \,, \tag{8.82}$$

is not a symmetry of θ-noncommutative space because the relations

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu} \tag{8.83}$$

are not compatible with Poincaré transformations. Indeed consider the standard representation of the Poincaré algebra on functions $h(x)$,

$$P_\mu(h) = i\partial_\mu(h)\ ,\ M_{\mu\nu}(h) = i(x_\mu\partial_\nu - x_\nu\partial_\mu)(h)\ , \tag{8.84}$$

then we have $M_{\rho\sigma}(\theta^{\mu\nu}) = 0$ while $M_{\rho\sigma}(x^\mu \star x^\nu - x^\nu \star x^\mu) \neq 0$. This is so because we use the undeformed Leibniz rule $M_{\rho\sigma}(x^\mu \star x^\nu - x^\nu \star x^\mu) = M_{\rho\sigma}(x^\mu) \star x^\nu + x^\mu \star M_{\rho\sigma}(x^\nu)$. In other words the Hopf algebra $U(iso(3,1))$ generated by the Poincaré Lie algebra and with usual coproducts

$$\Delta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu \quad , \quad \Delta(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \tag{8.85}$$

is not a symmetry of noncommutative spacetime.

One approach to overcome this problem is to just deform the coproduct Δ into the new coproduct $\Delta^{\mathscr{F}}(M_{\mu\nu}) = \mathscr{F}\Delta(M_{\mu\nu})\mathscr{F}^{-1}$ (see next section).

Another approach is to observe first that the action of $M_{\rho\sigma}$ on $h \star g$ is hybrid, indeed it mixes ordinary products with $\star$-products: $M_{\mu\nu}(h \star g) = ix_\mu\partial_\nu(h \star g) - ix_\nu\partial_\mu(h \star g)$. This is cured by considering a different action of the generators P_μ and $M_{\mu\nu}$ on noncommutative spacetime. The $\mathscr{L}^\star$ action defined in (8.47), accordingly with the general prescription (8.18), exactly replaces the ordinary product with the $\star$-product. For any function $h(x)$ we have,

$$\begin{aligned} \mathscr{L}^\star_{P_\mu}(h) &= i\partial_\mu(h) \\ \mathscr{L}^\star_{M_{\mu\nu}}(h) &= ix_\mu \star \partial_\nu(h) - ix_\nu \star \partial_\mu(h)\ . \end{aligned} \tag{8.86}$$

This action of the Poincaré generators on functions can be extended to an action of the universal enveloping algebra $U(iso(3,1))$ if $U(iso(3,1))$ is endowed with the new $\star$-product

$$\begin{aligned} \xi \star \zeta &:= \bar{\mathrm{f}}^\alpha(\xi)\bar{\mathrm{f}}_\alpha(\zeta) \\ &= \sum \frac{1}{n!}\left(\frac{-i}{2}\right)^n \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}[P_{\rho_1}\ldots[P_{\rho_n},\xi]...]\,[P_{\sigma_1}\ldots[P_{\sigma_n},\zeta]...]\,, \end{aligned} \tag{8.87}$$

for all ξ and ζ in $U(iso(3,1))$. For example, it is easy to see that

$$\mathscr{L}^\star_{M_{\mu\nu}\star M_{\rho\sigma}}(h) = \mathscr{L}^\star_{M_{\mu\nu}}(\mathscr{L}^\star_{M_{\rho\sigma}}(h))\ . \tag{8.88}$$

In formula (8.87) we have identified the Lie algebra of partial derivatives with the Lie algebra of momenta P_μ, so that

$$\mathscr{F} = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}, \quad \mathscr{R}^{-1} = e^{i\theta^{\mu\nu}P_\mu \otimes P_\nu}\ . \tag{8.89}$$

This identification is uniquely fixed by the representation (8.84): $P_\mu = i\partial_\mu$. Since products of the generators P_μ and $M_{\mu\nu}$ can be rewritten as sum of $\star$-products via the formula $\xi\zeta = \mathrm{f}^\alpha(\xi)\star\mathrm{f}_\alpha(\zeta)$, the elements P_μ and $M_{\mu\nu}$ generate the algebra $U_\star(iso(3,1))$.

The coproduct compatible with noncommutative spacetime is inferred from the Leibniz rule

$$\begin{aligned} x_\mu\star\partial_\nu(h\star g) &= x_\mu\star\partial_\nu(h)\star g + x_\mu\star h\star\partial_\nu(g) \\ &= x_\mu\star\partial_\nu(h)\star g + \overline{R}^\alpha(h)\star\overline{R}_\alpha(x_\mu)\star\partial_\nu(g)\ . \end{aligned} \tag{8.90}$$

The coproduct that implements this Leibniz rule is (cf. (8.44))

$$\Delta_\star(M_{\mu\nu}) = M_{\mu\nu}\otimes 1 + \overline{R}^\alpha\otimes\overline{R}_\alpha(M_{\mu\nu})\ . \tag{8.91}$$

Explicitly the coproduct on the generators P_μ and $M_{\mu\nu}$ reads

$$\begin{aligned} \Delta_\star(P_\mu) &= P_\mu\otimes 1 + 1\otimes P_\mu\ , \\ \Delta_\star(M_{\mu\nu}) &= M_{\mu\nu}\otimes 1 + 1\otimes M_{\mu\nu} + i\theta^{\alpha\beta}P_\alpha\otimes[P_\beta, M_{\mu\nu}]\ . \end{aligned} \tag{8.92}$$

The counit and antipode on the generators can be calculated from (8.45) and (8.46), they are given by

$$\begin{aligned} &\varepsilon(P_\mu) = \varepsilon(M_{\mu\nu}) = 0\ , \\ &S_\star(P_\mu) = -P_\mu\ ,\quad S_\star(M_{\mu\nu}) = -M_{\mu\nu} - i\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}]P_\sigma\ . \end{aligned} \tag{8.93}$$

We have constructed the Hopf algebra $U_\star(iso(3,1))$.

We recall that there are three natural conditions that the $\star$-Poincaré Lie algebra $iso_\star(3,1)$ has to satisfy. It has to be a linear subspace of $U_\star(iso(3,1))$ such that if $\{t_i\}_{i=1,\ldots,n}$ is a basis of $iso_\star(3,1)$, we have (sum understood on repeated indices)

$$\begin{aligned} i)&\quad \{t_i\} \text{ generates } U_\star(iso(3,1)) \\ ii)&\quad \Delta_\star(t_i) = t_i\otimes 1 + f_i{}^j\otimes t_j \\ iii)&\quad [t_i, t_j]_\star = C_{ij}{}^k t_k \end{aligned}$$

where $C_{ij}{}^k$ are structure constants and $f_i{}^j\in U_\star(iso(3,1))$ $(i,j=1,\ldots,n)$. In the last line the bracket $[\ ,\ ]_\star$ is the adjoint action (we use the notation $\Delta_\star(t) = t_{1_\star}\otimes t_{2_\star}$):

$$[t,t']_\star := ad^\star_t\, t' = t_{1_\star}\star t'\star S_\star(t_{2_\star})\ . \tag{8.94}$$

We have seen that the elements P_μ and $M_{\mu\nu}$ generate $U_\star(iso(3,1))$. They are deformed *infinitesimal* generators because they satisfy the Leibniz rule *ii*) and because they close under the adjoint action *iii*). In order to prove property *iii*) we perform a short calculation and obtain the explicit expression of the adjoint action (8.94),

$$[P_\mu, P_\nu]_\star = [P_\mu, P_\nu] \,,$$
$$[P_\rho, M_{\mu\nu}]_\star = [P_\rho, M_{\mu\nu}] = -[M_{\mu\nu}, P_\rho]_\star \,,$$
$$[M_{\mu\nu}, M_{\rho\sigma}]_\star = M_{\mu\nu} \star M_{\rho\sigma} - M_{\rho\sigma} \star M_{\mu\nu} - i\theta^{\alpha\beta}[P_\alpha, M_{\rho\sigma}][P_\beta, M_{\mu\nu}] = [M_{\mu\nu}, M_{\rho\sigma}] \,.$$

Notice that this result shows that the adjoint action (8.94) equals the deformed commutator

$$t \star t' - \overline{R}^\alpha(t') \star \overline{R}_\alpha(t) \,.$$

Property *iii*), i.e., closure under the adjoint action, explicitly reads

$$\begin{aligned} [P_\mu, P_\nu]_\star &= 0 \,, \\ [P_\rho, M_{\mu\nu}]_\star &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \,, \\ [M_{\mu\nu}, M_{\rho\sigma}]_\star &= -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \,. \end{aligned} \tag{8.95}$$

We notice that the structure constants are the same as in the undeformed case; however, the adjoint action $[M_{\mu\nu}, M_{\rho\sigma}]_\star$ is not the commutator $M_{\mu\nu} \star M_{\rho\sigma} - M_{\rho\sigma} \star M_{\mu\nu}$ anymore, it is a deformed commutator quadratic in the generators and $\star$-antisymmetric.

From (8.95) we immediately obtain the $\star$-Jacobi identities:

$$[t, [t', t'']_\star]_\star + [t', [t'', t]_\star]_\star + [t'', [t, t']_\star]_\star = 0 \,, \tag{8.96}$$

for all $t, t', t'' \in iso_\star(3,1)$.

It can be proven that the Hopf algebra $U_\star(iso(3,1))$ is the algebra freely generated by P_μ and $M_{\mu\nu}$ (we denote the product by $\star$) modulo the relations *iii*).

Note 8.5 In [32] we studied quantum Poincaré groups (in any dimension) obtained via abelian twists $\mathscr{F}$ different from the one considered here. Their Lie algebra is described according to *i*), *ii*), *iii*) (see for example Eqs. (6.65), (7.36), (7.6) and (7.7) in the first paper in [32, 33]). Because of these three properties the Lie algebra defines a differential calculus on the quantum Poincaré group manifold that respects the quantum Poincaré symmetry, (i.e., that is bicovariant).

8.3.2 Twisted Poincaré algebra

The Poincaré Hopf algebra $U^{\mathscr{F}}(iso(3,1))$ is another deformation of $U(iso(3,1))$, as algebras $U^{\mathscr{F}}(iso(3,1)) = U(iso(3,1))$; but $U^{\mathscr{F}}(iso(3,1))$ has the new coproduct

$$\Delta^{\mathscr{F}}(\xi) = \mathscr{F} \Delta(\xi) \mathscr{F}^{-1} \,, \tag{8.97}$$

for all $\xi \in U(iso(3,1))$. In Sect. 7.7 we wrote the explicit expression for $\Delta^{\mathscr{F}}(P_\mu)$ and $\Delta^{\mathscr{F}}(M_{\mu\nu})$, see Eqs. (7.93). The Hopf algebra $U^{\mathscr{F}}(iso(3,1))$ is the algebra generated by $M_{\mu\nu}$ and P_μ modulo the relations (8.81), and with coproduct (7.93) and counit and antipode that are as in the undeformed case (see the explicit expressions (7.94)). This Hopf algebra is a symmetry of noncommutative spacetime provided that we consider the "hybrid" action $M_{\mu\nu}(h\star g) = ix_\mu\partial_\nu(h\star g) - ix_\nu\partial_\mu(h\star g)$.

There is a canonical procedure in order to obtain the Lie algebra $iso^{\mathscr{F}}(3,1)$ of $U^{\mathscr{F}}(iso(3,1))$. We use the Hopf algebra isomorphism (8.73)

$$\begin{aligned} D\,:\, U_\star(iso(3,1)) &\to U^{\mathscr{F}}(iso(3,1)) \\ \xi &\mapsto \bar{\mathrm{f}}^\alpha(\xi)\bar{\mathrm{f}}_\alpha \end{aligned}$$

and define

$$iso^{\mathscr{F}}(3,1) := D(iso_\star(3,1)) \ .$$

The elements

$$P^{\mathscr{F}}_\mu := \bar{\mathrm{f}}^\alpha(P_\mu)\bar{\mathrm{f}}_\alpha = P_\mu \ , \tag{8.98}$$

$$\begin{aligned} M^{\mathscr{F}}_{\mu\nu} := \bar{\mathrm{f}}^\alpha(M_{\mu\nu})\bar{\mathrm{f}}_\alpha &= M_{\mu\nu} - \frac{i}{2}\theta^{\rho\sigma}[P_\rho, M_{\mu\nu}]P_\sigma \\ &= M_{\mu\nu} + \frac{1}{2}\theta^{\rho\sigma}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)P_\sigma \end{aligned} \tag{8.99}$$

are generators of the quantum Lie algebra $iso^{\mathscr{F}}(3,1)$. Their explicit quantum Lie algebra structure as well as their coproduct is given in Sect. 7.7.

8.4 Covariant derivative, torsion, and curvature

The noncommutative differential geometry set up in the previous sections allows to develop the formalism of covariant derivative, torsion, and curvature just by following the usual classical formalism.

On functions the covariant derivative equals the Lie derivative. Requiring that this holds in the $\star$-noncommutative case as well we immediately know the action of the $\star$-covariant derivative on functions, and in particular the Leibniz rule it has to satisfy. More in general we define the $\star$-covariant derivative $\nabla^\star_u$ along the vector field $u \in \Xi_\star$ to be the linear map $\nabla^\star_u : \Xi_\star \to \Xi_\star$ such that for all $u,v,z \in \Xi_\star$, $h \in A_\star$:

$$\nabla^\star_{u+v} z = \nabla^\star_u z + \nabla^\star_v z \ , \tag{8.100}$$

$$\nabla^\star_{h\star u} v = h \star \nabla^\star_u v \ , \tag{8.101}$$

$$\nabla^\star_u(h\star v) = \mathscr{L}^\star_u(h)\star v + \bar{R}^\alpha(h)\star\nabla^\star_{\bar{R}_\alpha(u)} v \ . \tag{8.102}$$

Notice that in the last line we have used the coproduct formula (8.44), $\Delta_\star(u) = u\otimes 1 + \overline{R}^\alpha \otimes \overline{R}_\alpha(u)$. Epression (8.102) is well defined because $\overline{R}_\alpha(u)$ is again a vector field.

The (noncommutative) connection coefficients ${\Gamma_{\mu\nu}}^\sigma$ are given by

$$\nabla^\star_\mu \partial_\nu = {\Gamma_{\mu\nu}}^\sigma \star \partial_\sigma = {\Gamma_{\mu\nu}}^\sigma\, \partial_\sigma \ , \tag{8.103}$$

where $\nabla^\star_\mu = \nabla^\star_{\partial_\mu}$. They uniquely determine the connection, indeed let $z = z^\mu \star \partial_\mu$, $u = u^\nu \star \partial_\nu$, then

$$\begin{aligned}\nabla^\star_z u &= z^\mu \star \nabla^\star_\mu (u^\nu \star \partial_\nu)\\ &= z^\mu \star \partial_\mu (u^\nu)\,\partial_\nu + z^\mu \star u^\nu \star \nabla^\star_\mu \partial_\nu\\ &= z^\mu \star \partial_\mu (u^\nu)\,\partial_\nu + z^\mu \star u^\nu \star {\Gamma_{\mu\nu}}^\sigma\,\partial_\sigma \ ;\end{aligned} \tag{8.104}$$

these equalities are equivalent to the connection properties (8.101) and (8.102).

The covariant derivative is extended to tensor fields using the deformed Leibniz rule

$$\nabla^\star_u (v\otimes_\star z) = \nabla^\star_u(v)\otimes_\star z + \overline{R}^\alpha(v)\otimes_\star \nabla^\star_{\overline{R}_\alpha(u)} z \ .$$

Requiring compatibility of the covariant derivative with the contraction operator gives the covariant derivative on 1-forms, we have $\nabla^\star_z = z^\mu \star \nabla^\star_\mu$, and

$$\nabla^\star_\mu(\omega_\rho dx^\rho) = \partial_\mu(\omega_\rho)\, dx^\rho - {\Gamma_{\mu\rho}}^\nu \star \omega_\nu\, dx^\rho \ . \tag{8.105}$$

The torsion T and the curvature R associated with a connection $\nabla^\star$ are the linear maps $\mathsf{T} : \Xi_\star \times \Xi_\star \to \Xi_\star$ and $\mathsf{R}^\star : \Xi_\star \times \Xi_\star \times \Xi_\star \to \Xi_\star$ defined by

$$\mathsf{T}(u,v) := \nabla^\star_u v - \nabla^\star_{\overline{R}^\alpha(v)} \overline{R}_\alpha(u) - [u\, v]_\star \ , \tag{8.106}$$

$$\mathsf{R}(u,v,z) := \nabla^\star_u \nabla^\star_v z - \nabla^\star_{\overline{R}^\alpha(v)} \nabla^\star_{\overline{R}_\alpha(u)} z - \nabla^\star_{[u\,v]_\star} z \ , \tag{8.107}$$

for all $u,v,z \in \Xi_\star$. From the antisymmetry property of the bracket $[\ ]_\star$, see (8.67), it easily follows that the torsion T and the curvature R have the following $\star$-antisymmetry property

$$\begin{aligned}\mathsf{T}(u,v) &= -\mathsf{T}(\overline{R}^\alpha(v), \overline{R}_\alpha(u)) \ ,\\ \mathsf{R}(u,v,z) &= -\mathsf{R}(\overline{R}^\alpha(v), \overline{R}_\alpha(u), z) \ .\end{aligned}$$

The presence of the R-matrix in the definition of torsion and curvature ensures that T and R are left $A_\star$-linear maps [16], i.e.,

$$\mathsf{T}(f\star u, v) = f\star \mathsf{T}(u,v) \quad , \quad \mathsf{T}(\partial_\mu, f\star v) = f\star \mathsf{T}(\partial_\mu, v)$$

(for any ∂_μ), and similarly for the curvature. We have seen that any left $A_\star$-linear map $\Xi_\star \to A_\star$ is identified with a tensor, and precisely a 1-form (recall comments after (8.37)). Similarly the $A_\star$-linearity of T and R ensures that we have well defined the torsion tensor and the curvature tensor.

One can also prove (twisted) first and second Bianchi identities [16, 27].

The coefficients ${\mathsf{T}_{\mu\nu}}^\rho$ and ${\mathsf{R}_{\mu\nu\rho}}^\sigma$ with respect to the partial derivatives basis $\{\partial_\mu\}$ are defined by

$$\mathsf{T}(\partial_\mu,\partial_\nu) = {\mathsf{T}_{\mu\nu}}^\rho\partial_\rho, \qquad \mathsf{R}(\partial_\mu,\partial_\nu,\partial_\rho) = {\mathsf{R}_{\mu\nu\rho}}^\sigma\partial_\sigma, \tag{8.108}$$

and they explicitly read

$$\begin{aligned} {\mathsf{T}_{\mu\nu}}^\rho &= {\Gamma_{\mu\nu}}^\rho - {\Gamma_{\nu\mu}}^\rho \;\;, \\ {\mathsf{R}_{\mu\nu\rho}}^\sigma &= \partial_\mu {\Gamma_{\nu\rho}}^\sigma - \partial_\nu {\Gamma_{\mu\rho}}^\sigma + {\Gamma_{\nu\rho}}^\beta \star {\Gamma_{\mu\beta}}^\sigma - {\Gamma_{\mu\rho}}^\beta \star {\Gamma_{\nu\beta}}^\sigma \;. \end{aligned} \tag{8.109}$$

As in the commutative case the Ricci tensor is a contraction of the curvature tensor,

$$\mathsf{Ric}_{\mu\nu} = {\mathsf{R}_{\rho\mu\nu}}^\rho. \tag{8.110}$$

A definition of the Ricci tensor that is independent from the $\{\partial_\mu\}$ basis is also possible [16].

8.5 Metric and Einstein equations

In order to define a $\star$-metric we need to define $\star$-symmetric elements in $\Omega_\star \otimes_\star \Omega_\star$. Recalling the $\star$-antisymmetry of the wedge $\star$-product (8.30) we see that $\star$-symmetric elements are of the form

$$\omega \otimes_\star \omega' + \overline{R}^\alpha(\omega') \otimes_\star \overline{R}_\alpha(\omega) \;. \tag{8.111}$$

In particular any symmetric tensor in $\Omega \otimes \Omega$,

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \;, \tag{8.112}$$

$g_{\mu\nu} = g_{\nu\mu}$, is also a $\star$-symmetric tensor in $\Omega_\star \otimes_\star \Omega_\star$ because

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = g_{\mu\nu} \star dx^\mu \otimes_\star dx^\nu, \tag{8.113}$$

and the action of the R-matrix is the trivial one on dx^ν. We denote by $g^{\star\mu\nu}$ the star inverse of $g_{\mu\nu}$,

$$g^{\star\mu\rho} \star g_{\rho\nu} = g_{\nu\rho} \star g^{\star\rho\mu} = \delta^\mu_\nu \;. \tag{8.114}$$

The metric $g_{\mu\nu}$ can be expanded order by order in the noncommutative parameter $\theta^{\rho\sigma}$. Any commutative metric is also a noncommutative metric, indeed the $\star$-inverse metric can be constructed order by order in the noncommutativity parameter (see also Sect. 3.4). Contrary to [9, 52], we see that in our approach there are infinitely many metrics compatible with a given noncommutative differential geometry, noncommutativity does not single out a preferred metric.

A connection that is metric compatible is a connection that for any vector field u satisfies, $\nabla^\star_u g = 0$, this is equivalent to the equation

$$\nabla^\star_\mu g_{\rho\sigma} - \Gamma_{\mu\rho}{}^{\nu} \star g_{\nu\sigma} - \Gamma_{\mu\sigma}{}^{\nu} \star g_{\rho\nu} = 0\ . \tag{8.115}$$

We permute the indices in this expression, use the symmetry $g_{\mu\nu} = g_{\nu\mu}$, and add the corresponding equations to obtain

$$\Gamma_{\mu\nu}{}^{\rho} \star g_{\rho\sigma} = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{8.116}$$

We therefore obtain that there is a unique torsion-free metric-compatible connection. It is given by

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \star g^{\star\sigma\rho}. \tag{8.117}$$

We now construct the curvature tensor and the Ricci tensor using this uniquely defined connection. Finally the noncommutative version of Einstein equations (in vacuum) is

$$\mathsf{Ric}_{\mu\nu} = 0, \tag{8.118}$$

where the dynamical field is the metric g.

Appendix

8.6 Differential operators and vector fields

We briefly describe the algebra of differential operators and show that it is not a Hopf algebra by relating it to the Hopf algebra of vector fields.

Differential operators on the space of functions $A = Fun(\mathbb{R}^4)$ are elements of the form $f(x)^{\mu_1\cdots\mu_n}\partial_{\mu_1}\ldots\partial_{\mu_n}$. They form an algebra, the only nontrivial commutation relations are between functions and partial derivatives,

$$\partial_\mu f = \partial_\mu(f) + f\partial_\mu\ , \tag{8.119}$$

where both ∂_μ and f act on functions (the action of f on the function h is given by the product fh). Differential operators of zeroth order are functions. Differential operators of first order $Diff^1$ are derivations of the algebra A of functions (i.e., they satisfy the Leibniz rule); they are therefore vector fields Ξ (infinitesimal local diffeomorphisms[4]). The isomorphism between vector fields and first-order differential operators is given by the Lie derivative

$$\begin{aligned}\mathscr{L} : \Xi &\to \mathit{Diff}^1 \\ v &\mapsto \mathscr{L}_v, \end{aligned} \tag{8.120}$$

where

$$\mathscr{L}_v(f) = v(f) \; .$$

We use the notation $\mathscr{L}_v$ in order to stress that the abstract Lie algebra element $v \in \Xi$ is seen as a differential operator. The Lie derivative can be extended to a map from the universal enveloping algebra of vector fields $U\Xi$ to all differential operators

$$\begin{aligned}\mathscr{L} : \Xi &\to \mathit{Diff} \\ uv...z &\mapsto \mathscr{L}_u \mathscr{L}_v ... \mathscr{L}_z. \end{aligned} \tag{8.121}$$

Notice that on the left-hand side the product uv is in $U\Xi$ (recall the paragraph after (8.40)), while on the right-hand side the product $\mathscr{L}_u \mathscr{L}_v = \mathscr{L}_u \circ \mathscr{L}_v$ is the usual composition product of operators.

The map $\mathscr{L}$ is an algebra morphism between the algebras $U\Xi$ and *Diff*. It is not surjective because the image of $U\Xi$ does not contain the full space of functions A but only the constant ones (the multiples of the unit of $U\Xi$).

In order to show that the map $\mathscr{L} : U\Xi \to \mathit{Diff}$ is not injective we consider the vector fields

$$\begin{aligned} u = f\partial_\mu \quad &, \quad v = \partial_\nu \; , \\ u' = f\partial_\nu \quad &, \quad v' = \partial_\mu \; , \end{aligned}$$

where for example $f = x^\nu$, and we show that

$$uv \neq u'v' \qquad \text{in } U\Xi \, . \tag{8.122}$$

The map $\mathscr{L}$ is then not injective because $f\partial_\mu(\partial_\nu(h)) = f\partial_\nu(\partial_\mu(h))$ for any function h implies

$$\mathscr{L}_{uv} = \mathscr{L}_{u'v'} \; .$$

The algebra $U\Xi$ is a Hopf algebra, in particular there is a well-defined coproduct map Δ, and therefore one way to prove the inequality (8.122) is to prove that $\Delta(uv) \neq \Delta(u'v')$. We calculate

$$\begin{aligned}\Delta(uv) &= \Delta(u)\Delta(v) = uv \otimes 1 + u \otimes v + v \otimes u + 1 \otimes uv \\ &= f\partial_\mu \partial_\nu \otimes 1 + f\partial_\mu \otimes \partial_\nu + \partial_\nu \otimes f\partial_\mu + 1 \otimes f\partial_\mu \partial_\nu \; , \end{aligned}$$

[4] Infinitesimal diffeomorphisms correspond to complete vector fields. If the manifold is not compact vector fields are not necessarily complete, then they only give rise to a local one parameter group of local diffeomorphisms.

and

$$\Delta(u'v') = f\partial_\nu\partial_\mu \otimes 1 + f\partial_\nu \otimes \partial_\mu + \partial_\mu \otimes f\partial_\nu + 1 \otimes f\partial_\nu\partial_\mu \ . \tag{8.123}$$

These two expressions are different. For example, by applying the product map in $U\Xi$, $\cdot : U\Xi \otimes U\Xi \to U\Xi$ and then the map $\mathscr{L} : U\Xi \to \mathit{Diff}$ we, respectively, obtain

$$3f\partial_\mu\partial_\nu + \partial_\nu f\partial_\mu \neq 3f\partial_\nu\partial_\mu + \partial_\mu f\partial_\nu \ . \tag{8.124}$$

From this proof we conclude that we cannot equip the algebra of differential operators Diff with a coproduct like the one in $U\Xi$. The map defined by $\Delta(\mathscr{L}_u) = \mathscr{L}_u \otimes 1 + 1 \otimes \mathscr{L}_u$ and extended multiplicatively to all Diff is not well defined because $\mathscr{L}_{uv} = \mathscr{L}_{u'v'}$, while

$$\Delta(\mathscr{L}_{uv}) = \Delta(\mathscr{L}_u)\Delta(\mathscr{L}_v) \neq \Delta(\mathscr{L}_{u'})\Delta(\mathscr{L}_{v'}) = \Delta(\mathscr{L}_{u'v'}) \ ,$$

as is easily seen by applying the product in Diff, $\circ : \mathit{Diff} \otimes \mathit{Diff} \to \mathit{Diff}$ (we obtain again (8.124)).

8.7 Proof that the coproduct $\Delta_\star$ is coassociative

We have to prove that

$$(\Delta_\star \otimes id)\Delta_\star(u) = (id \otimes \Delta_\star)\Delta_\star(u) \ .$$

The left-hand side explicitly reads

$$\begin{aligned}(\Delta_\star \otimes id)\Delta_\star(u) &= (\Delta_\star \otimes id)(u \otimes 1 + \overline{R}^\alpha \otimes \overline{R}_\alpha(u)) \\ &= u \otimes 1 \otimes 1 + \overline{R}^\beta \otimes \overline{R}_\beta(u) \otimes 1 + \Delta_\star(\overline{R}^\alpha) \otimes \overline{R}_\alpha(u) \ .\end{aligned}$$

The right-hand side is

$$\begin{aligned}(id \otimes \Delta_\star)\Delta_\star(u) &= u \otimes \Delta_\star(1) + \overline{R}^\alpha \otimes \Delta_\star(\overline{R}_\alpha(u)) \\ &= u \otimes 1 \otimes 1 + \overline{R}^\alpha \otimes \overline{R}_\alpha(u) \otimes 1 + \overline{R}^\alpha \otimes \overline{R}^\gamma \otimes \overline{R}_\gamma\overline{R}_\alpha(u) \ .\end{aligned}$$

These two expressions coincide because

$$\begin{aligned}\Delta_\star(\overline{R}^\alpha) \otimes \overline{R}_\alpha &= e^{-i\theta^{\mu\nu}\Delta_\star(\partial_\mu)\otimes\partial_\nu} = e^{-i\theta^{\mu\nu}(\partial_\mu\otimes 1\otimes\partial_\nu + 1\otimes\partial_\mu\otimes\partial_\nu)} \\ &= \overline{R}^\alpha \otimes \overline{R}^\gamma \otimes \overline{R}_\gamma\overline{R}_\alpha \ .\end{aligned} \tag{8.125}$$

8.8 Proof that the bracket $[u\ v]_\star$ is the adjoint action

We have to prove that

$$[u\ v]_\star = ad^\star_u v\ .$$

We know that the bracket $[u\ v]_\star$ equals the deformed commutator

$$[u\ v]_\star = u \star v - \overline{R}^\alpha(v) \star \overline{R}_\alpha(u)\ .$$

On the other hand, the adjoint action reads

$$\begin{aligned} ad^\star_u v = u_{1_\star} \star v \star S_\star(u_{2_\star}) &= u \star v + \overline{R}^\alpha \star v \star S_\star(\overline{R}_\alpha(u)) \\ &= u \star v - \overline{R}^\alpha \star v \star \overline{R}^\beta(\overline{R}_\alpha(u))\overline{R}_\beta\ . \end{aligned}$$

Now the property

$$\partial_\mu \star v = \partial_\mu v = \partial_\mu(v) + v\,\partial_\mu\ , \tag{8.126}$$

that using the coproduct $\Delta_\star(\partial_\mu) \equiv \partial_{\mu_{1_\star}} \otimes \partial_{\mu_{2_\star}} = \partial_\mu \otimes 1 + 1 \otimes \partial_\mu$ can be written as

$$\partial_\mu \star v = \partial_\mu v = \partial_{\mu_{1_\star}}(v)\,\partial_{\mu_{2_\star}}\ ,$$

implies

$$\overline{R}^\alpha \star v = \overline{R}^\alpha v = \overline{R}^\alpha_{1_\star}(v)\,\overline{R}^\alpha_{2_\star}\ .$$

The coproduct formula (8.125) then implies

$$\begin{aligned} \overline{R}^\alpha \star v \star \overline{R}^\beta(\overline{R}_\alpha(u))\overline{R}_\beta &= \overline{R}^\alpha(v)\overline{R}^\gamma \star \overline{R}^\beta((\overline{R}_\gamma\overline{R}_\alpha)(u))\overline{R}_\beta \\ &= \overline{R}^\alpha(v) \star \overline{R}^\beta((\overline{R}_\gamma\overline{R}_\alpha)(u))\overline{R}_\beta\overline{R}^\gamma \\ &= \overline{R}^\alpha(v) \star (\overline{R}^\beta\overline{R}_\gamma\overline{R}_\alpha)(u)\overline{R}_\beta\overline{R}^\gamma \\ &= \overline{R}^\alpha(v) \star \overline{R}_\alpha(u), \end{aligned}$$

where in the second equality we iterated property (8.126) (with $\overline{R}^\beta((\overline{R}_\gamma\overline{R}_\alpha)(u))\overline{R}_\beta$ instead of v) and used the antisymmetry of $\theta^{\mu\nu}$ in order to cancel the first addend in (8.126). In the last equality we used that $\overline{R}^\beta\overline{R}_\gamma \otimes \overline{R}_\beta\overline{R}^\gamma = \mathscr{R}^{-1}\mathscr{R} = 1 \otimes 1$ because of the antisymmetry of $\theta^{\mu\nu}$.

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Chapter 9
Twist Deformations of Quantum Integrable Spin Chains

Petr Kulish

Twist deformations of spacetime lead to deformed field theories with twisted symmetries. Twisted symmetries are quantum group symmetries. Most integrable spin systems have dynamical symmetries related to appropriate quantum groups. We discuss the changes of the properties of these systems under twist transformations of quantum groups. A main example is the isotropic Heisenberg spin chain and the jordanian twist of the universal enveloping algebra of $sl(2)$. It is shown that the spectrum of the XXX spin chain is preserved under the twist deformation while the structure of the eigenstates depends on the choice of boundary conditions. Another example is provided by abelian twists, these give physical deformations of closed spin chains corresponding to higher rank Lie algebras, e.g., $gl(n)$. The energy spectrum of these integrable models is changed and correspondingly their eigenvectors.

9.1 Introduction

One of the cornerstone of the quantum inverse scattering method was the isotropic Heisenberg spin chain [1] exactly solved by H. Bethe [2]. The development of the quantum inverse scattering method (QISM) [3–7], as an approach to the construction and solution of quantum integrable systems, has led to the foundations of the theory of quantum groups [8–11]. Both in QISM and in quantum groups a fundamental, defining object is the R-matrix. V. Drinfel'd introduced an important transformation of quantum groups: a twist of coproduct map. The R-matrix is changed under Drinfel'd twist transformations. We would like to discuss the corresponding changes in integrable models taking as examples the isotropic XXX (9.1) and the anisotropic XXZ spin chains (9.20). These systems are more elementary than the field theories on noncommutative spaces discussed in the previous chapters. The aim is to see what kinds of modifications on these physical systems are produced by twisting their underlying symmetry structures.

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To explain magnetic properties of solids in quantum theory a model of interacting half-integer spins was proposed by W. Heisenberg in 1928 [1]. The hamiltonian of the isotropic model (XXX spin chain) is given in terms of Pauli sigma matrices $\sigma_k^\alpha, \alpha = x,y,z$; at each site $k = 1,2,\dots,N$ of a one-dimensional chain

$$H_{XXX} = \sum_{m=1}^{N} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \left(\sigma_m^z \sigma_{m+1}^z - 1\right)\right) - \frac{h}{2} \sum_{m=1}^{N} \sigma_m^z. \tag{9.1}$$

The following periodic ($\kappa = 1$) or quasi-periodic ($\kappa \neq 1$) boundary conditions are imposed

$$\sigma_{N+1}^z = \sigma_1^z, \qquad \sigma_{N+1}^\pm = \kappa^{\pm 1} \sigma_1^\pm, \qquad \sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y).$$

(Often the quasi-periodic boundary conditions are referred to as the twisted ones, but the word “twist” is reserved in this book for the theory of quantum groups.) The hamiltonian in (9.1) is an operator in the Hilbert space of spin states

$$\mathscr{H} = \bigotimes_{m=1}^{N} \mathbb{C}_m^2,$$

which is the tensor product of the two-dimensional Hilbert spaces associated with each site of the chain $m = 1,2,\dots,N$. The explicit form of these sigma matrices $\sigma^\alpha, \alpha = x,y,z$,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9.2}$$

enables one to write the hamiltonian density for zero magnetic field $h = 0$ as a permutation operator $\mathscr{P}_{m\,m+1}$ of neighboring spaces $\mathbb{C}_m^2 \otimes \mathbb{C}_{m+1}^2$: $\mathscr{P}(v \otimes w) = w \otimes v$, where $v, w \in \mathbb{C}^2$. Indeed

$$\sum_\alpha \sigma_m^\alpha \sigma_{m+1}^\alpha = 2\mathscr{P}_{m\,m+1} - I_{m\,m+1}\ ; \tag{9.3}$$

here $I_{m\,m+1}$ is the identity matrix. Taking the basis vectors of $\mathbb{C}^2$ as $e^{(+)} = \binom{1}{0}, e^{(-)} = \binom{0}{1}$, so that $\sigma^\pm e^{(\pm)} = 0$, and the basis vectors of $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$e^{(+)} \otimes e^{(+)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^{(+)} \otimes e^{(-)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$e^{(-)} \otimes e^{(+)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^{(-)} \otimes e^{(-)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{9.4}$$

the permutation (flip) matrix is

$$\mathscr{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{9.5}$$

In terms of the permutation operators $\mathscr{P}_{mm+1}$ the hamiltonian in (9.1), that from now on we consider with zero magnetic field $h = 0$, reads

$$H_{XXX} = 2 \sum_{m=1}^{N} (\mathscr{P}_{mm+1} - I_{mm+1}) . \tag{9.6}$$

H_{XXX} is an element of the group algebra $\mathbb{C}[\mathscr{S}_N]$ of the symmetric group $\mathscr{S}_N$ (the group of permutations of N objects). See (7.52) for the definition of group algebra $\mathbb{C}[\mathscr{S}_N]$. One can rewrite this hamiltonian using raising σ^+ and lowering σ^- matrices,

$$H_{XXX} = 2 \sum_{m=1}^{N} \left(\sigma_m^+ \sigma_{m+1}^- + \sigma_m^- \sigma_{m+1}^+ + \frac{1}{2}(\sigma_m^z \sigma_{m+1}^z - 1) \right) . \tag{9.7}$$

Then it is easy to see that the tensor product state

$$\Omega = \bigotimes_{k=1}^{N} e_k^{(+)} = \bigotimes_{k=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_k \tag{9.8}$$

is an eigenvector of H_{XXX} with zero eigenvalue Ω,

$$H_{XXX}\, \Omega = 0.$$

This state Ω corresponds to all spins up, and it is called the ferromagnetic state.

The complete spectrum of the energy operator H_{XXX} and its eigenvectors were found by H. Bethe in 1931 [2]. Due to the obvious rotational invariance H_{XXX} commutes with the generators of rotations (global spin):

$$[H_{XXX}, S^\alpha] = 0, \quad S^\alpha = \frac{1}{2} \sum_{k=1}^{N} \sigma_k^\alpha, \quad [S^\alpha, S^\beta] = i\varepsilon^{\alpha\beta\gamma} S^\gamma. \tag{9.9}$$

Hence, the Hilbert space of states $\mathscr{H} = \bigotimes_1^N \mathbb{C}^2$ can be decomposed into invariant subspaces with fixed value of the third component S^z:

$$\mathscr{H} = \bigotimes_{1}^{N} \mathbb{C}^2 = \bigoplus_{M=0}^{N} \mathscr{H}_{\frac{1}{2}N-M}. \tag{9.10}$$

Consider the shift operator U

$$U = \mathscr{P}_{1N} \dots \mathscr{P}_{13} \mathscr{P}_{12} =: \prod_{k=1}^{N-1} \mathscr{P}_{1k+1}, \qquad U \sigma_k^{\alpha} = \sigma_{k+1}^{\alpha} U\,. \tag{9.11}$$

It commutes with the hamiltonian

$$[H_{XXX}, U] = 0\,. \tag{9.12}$$

Then it is easy to see that the one-magnon state

$$\Psi(z) = \sum_{k=1}^{N} z^k \sigma_k^- \Omega \tag{9.13}$$

is a common eigenvector of H_{XXX} and U,

$$U\Psi(z) = z^{-1}\Psi(z), \quad H_{XXX}\Psi(z) = 2(z + z^{-1} - 2)\Psi(z), \tag{9.14}$$

provided that the quasimomentum z satisfies the quantization condition

$$z^N = 1, \quad \log z = 2\pi i k/N, \quad k = 1, 2, \dots, N-1. \tag{9.15}$$

The module $|z|$ is equal to 1. Hence the magnon energy is negative, and to find the ground state with the lowest energy one needs to analyze states with many magnons.

Bethe's proposal was to search for eigenvectors of H_{XXX} in the form of the so-called (coordinate) Bethe ansatz: a linear combination of products of one-magnon states

$$\Psi(z_1, \dots, z_M) = \sum_{1 \le n_1 < n_2 < \dots < n_M \le N} \; \sum_{\pi \in \mathscr{S}_M} A\left(\pi, \{z_j\}_1^M\right) z_{\pi(1)}^{n_1} z_{\pi(2)}^{n_1} \dots z_{\pi(M)}^{n_M} \prod_{j=1}^{M} \sigma_{n_j}^- \, \Omega\,. \tag{9.16}$$

Here z_j are the quasimomenta of the M magnons, $\mathscr{S}_M$ is the symmetric group with $M!$ elements $\{\pi\}$, and $A(\pi, \{z_j\}_1^M)$ are the amplitudes depending on $\{z_j\}_1^M$ and π.

The description of a thermodynamic limit $N \to \infty$ corresponding to an infinite antiferromagnetic chain was given by L. Hulthen in 1938 [12].

The requirement is that the M magnon vector (9.16) is an eigenvector of H_{XXX}, and the spin chain periodicity condition results in the explicit form of the coefficients $A(\pi; \{z_j\}_1^M)$ and the quantization conditions of quasimomenta $\{z_j\}_1^M$ (the so-called Bethe equations):

$$z_j^N = \prod_{k \neq j}^{M} \frac{z_j z_k + 1 - 2z_j}{2z_k - z_j z_k - 1}, \quad j = 1, 2, \dots, M. \tag{9.17}$$

The corresponding energy is

$$E_M = \sum_{j=1}^{M} 2(z_j + {z_j}^{-1} - 2). \tag{9.18}$$

The factors on the RHS of (9.17) are scalar two-magnon scattering matrices $S(z_j,\ z_k) = S(z_k,\ z_j)^{-1}$. A detailed deduction of these relations can be found in monographs (e.g. [13–15]). We will obtain them using the QISM in the next section.

There is also a different parameterization λ of quasimomenta

$$z(\lambda) = \frac{\lambda + \eta/2}{\lambda - \eta/2},$$

which is more convenient for the QISM formalism, where λ is known also as a spectral parameter. Although by a scaling $\lambda \to \eta\lambda$ the parameter η can be omitted it is useful to preserve it for the future discussions, e.g., of the quasiclassical limit $\eta \to 0$. Usually, one puts $\eta = i$ to get real-valued λ for $|z| = 1$. The one-magnon energy in terms of λ and $\eta = i$ is $E(\lambda) = -4/(4\lambda^2 + 1)$, and the Bethe equations (9.17) in terms of λ reads as follows:

$$\left(\frac{\lambda_j + \eta/2}{\lambda_j - \eta/2}\right)^N = \prod_{k\neq j}^{M} \frac{\lambda_j - \lambda_k + \eta}{\lambda_j - \lambda_k - \eta}. \tag{9.19}$$

It is instructive to mention two obvious algebras related to the isotropic Heisenberg spin chain: the rotational symmetry Lie algebra $sl(2)$ of H_{XXX} (9.6), (9.9), and the group algebra $\mathbb{C}[\mathscr{S}_N]$ of the symmetric group $\mathscr{S}_N$. We already remarked that the expression of the hamiltonian density in terms of permutation operators (9.6) shows that $H_{XXX} \in \mathbb{C}[\mathscr{S}_N]$. There is also a much bigger dynamical symmetry algebra, the so-called Yangian $\mathscr{Y}(sl(2))$ [4] which includes all the observables of the model (see Sect. 9.2).

Similar solution using the coordinate Bethe ansatz was constructed by R. Orbach in 1958 [16] for the anisotropic Heisenberg spin chain

$$H_{XXZ} = \sum_{k=1}^{N} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta(\sigma_k^z \sigma_{k+1}^z - 1)\right), \tag{9.20}$$

where $\Delta \in (-\infty, \infty)$ is an anisotropy parameter. The only obvious symmetry of this spin chain is the $U(1)$ group with the Lie algebra generator S^z (9.9). The space of states is also decomposed according to the eigenvalues of S^z

$$\mathscr{H} = \bigoplus_{M=0}^{N} \mathscr{H}_{\frac{1}{2}N-M}. \tag{9.21}$$

However, under a minor modification of the XXZ model hamiltonian (concerning an appropriate boundary condition instead of the periodicity one, cf. Sect. 9.2) the symmetry algebra is "similar" to the $sl(2)$ one; it is the quantum algebra $\mathscr{U}_q(sl(2))$ with three generators [8] (see also Sect. 7.4). As a second algebra of this XXZ model one has the Hecke algebra $\mathscr{H}_N(q)$ instead of $\mathbb{C}[\mathscr{S}_N]$. Finally a dynamical symmetry algebra for this model is the quantum affine algebra $\mathscr{U}_q(\widehat{sl}(2))$ [17].

In the next section we solve the XXX and XXZ models by a pure algebraic approach using the quantum inverse scattering method (QISM). For this reason now we write down only the spectrum of H_{XXZ} and we consider the corresponding Bethe equations for the quasimomenta with a different parameterization $\{\mu_j\}_1^M$,

$$H_{XXZ}\Psi(\{\mu_j\}_1^M) = E_M(\{\mu_j\}_1^M)\Psi(\{\mu_j\}_1^M), \tag{9.22}$$

$$E_M(\{\mu_j\}_1^M) = \sum_{j=1}^{M} \frac{\Delta^2-1}{\Delta - \cos 2\mu_j} = \sum_{j=1}^{M} \frac{(\cosh\eta)^2-1}{\cosh\eta - \cos 2\mu_j}, \tag{9.23}$$

$$\left(\frac{\sinh(\mu_j+\frac{1}{2}\eta)}{\sinh(\mu_j-\frac{1}{2}\eta)}\right)^N = \prod_{k\neq j}^{M} \frac{\sinh(\mu_j-\mu_k+\eta)}{\sinh(\mu_j-\mu_k-\eta)}. \tag{9.24}$$

We are using the standard parameterization of the anisotropy parameter Δ,

$$\Delta = \frac{1}{2}(q+q^{-1}) = \cosh(\eta), \quad q = \exp(\eta). \tag{9.25}$$

We finish this introduction by recalling that integrable quantum spin chains are closely related to exactly solved models of statistical mechanics on square lattice (à la two-dimensional Ising model) [15]. The trace of the transfer matrix $t(\lambda)$, which is the generating function of the integrals of motion of the spin system, leads to the partition function Z of the corresponding lattice statistical model. The entries of the R-matrix, a fundamental object of the QISM, are the Boltzmann weights of the local configurations [13–15].

9.2 Algebraic Bethe ansatz (QISM)

In this section we review the QISM formalism. We obtain the eigenvectors (9.22), the eigenvalues (9.23), and the quantization conditions (9.24) of the XXZ model, and the corresponding ones of the XXX model, via an algebraic approach (algebraic Bethe ansatz). This algebraic method is analogous to the treatment à la Dirac of the quantum harmonic oscillator with creation and annihilation operators. We construct a particular transformation converting the variables σ_k^α into a new set of operators. More precisely the aim is to transform the original spin $1/2$ operators σ_k^α (that are local operators because they act only on the kth site) to a set of new nonlocal operators in $\mathscr{H}$ with peculiar algebraic properties independent from the number of sites N. We denote these nonlocal operators by $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$. The hamiltonian H_{XXX} is expressed in terms of these operators, and by acting on the vacuum state Ω with the creation operators $B(\lambda_j)$ we also construct its eigenstates. A similar scheme holds also for the XXZ model.

Next we briefly discuss the underlying dynamical symmetry algebras. These are the Yangian $\mathscr{Y}(sl(2))$ for the XXX model and the quantum affine algebra $\mathscr{U}_q(\widehat{sl(2)})$ for the XXZ model. Deformations of the XXX and XXZ models obtained by twisting of these dynamical symmetries are then discussed in Sect. 9.3.

9.2.1 QISM for the XXX model

The main object of the transformation from the local operators $\sigma_k \in \mathrm{End}(\mathscr{H})$ to the nonlocal ones $A(\lambda)$, $B(\lambda)$, $C(\lambda), D(\lambda) \in \mathrm{End}(\mathscr{H})$ is an auxiliary operator: the L-matrix. It is a 2×2 matrix on an auxiliary space. The matrix entries depend on the local observables σ_k^α at a given site k and on the spectral parameter λ. In the case of the XXX model the L-matrix is

$$L_{ak}(\lambda) = \lambda I + \frac{1}{2}\eta \sum_\alpha \sigma^\alpha \otimes \sigma_k^\alpha = \begin{pmatrix} \lambda+\eta/2 & 0 & 0 & 0 \\ 0 & \lambda-\eta/2 & \eta & 0 \\ 0 & \eta & \lambda-\eta/2 & 0 \\ 0 & 0 & 0 & \lambda+\eta/2 \end{pmatrix}. \tag{9.26}$$

The indices a and k refer to the auxiliary space the matrices σ^α act and to the quantum space $\mathbb{C}_k^2$ (a factor in the definition of $\mathscr{H}$). On the other factors of $\mathscr{H}$ the L_{ak}-matrix acts as the identity. The L-operator in (9.26) is written as a 4×4 matrix in $\mathbb{C}_a^2 \otimes \mathbb{C}_k^2$ and one can recognize the local operators σ_k^α as 2×2 blocks of the 4×4 matrix.

Using the L-operator (9.26) a new set of variables (operators in the Hilbert space $\mathscr{H}$ (9.10) depending on the parameter λ) is introduced by an ordered product of $L_{ak}(\lambda)$ as 2×2 matrices on the auxiliary space $\mathbb{C}_a^2$,

$$T(\lambda) := L_{aN}(\lambda) L_{aN-1}(\lambda) \ldots L_{a1}(\lambda).$$

This new operator $T(\lambda)$ is the QISM monodromy matrix [3–6]. It is a 2×2 matrix in the auxiliary space $V_a \simeq \mathbb{C}^2$. Its entries

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{9.27}$$

are operators in $\mathscr{H}$. They are the new nonlocal variables. The commutation relations of these new operators $A(\lambda), \ldots, D(\lambda) \in \mathrm{End}(\mathscr{H})$ can be obtained from the local relation for the L-operator at one site (see for example [6]):

$$R_{12}(\lambda - \mu) L_{1k}(\lambda) L_{2k}(\mu) = L_{2k}(\mu) L_{1k}(\lambda) R_{12}(\lambda - \mu), \tag{9.28}$$

where the R-matrix is found from the previous equation to be

$$R(\lambda) = \lambda I + \eta \mathscr{P}. \tag{9.29}$$

The R-matrix in (9.29) acts on the tensor product of two auxiliary spaces $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2$. Equation (9.28) involves operators on $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2 \otimes \mathbb{C}_k^2$, where $\mathbb{C}_k^2$ is the space of spin quantum states at site k. The operators R and L_{ak} are understood to act in $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2 \otimes \mathbb{C}_k^2$ via the embeddings

$$R_{12}(\lambda-\mu) = R(\lambda-\mu) \otimes 1 \;, \qquad L_{2k}(\mu) = 1 \otimes L_{ak}(\mu) \;,$$

and similarly $L_{1k}(\lambda)$ acts as $L_{ak}(\lambda)$ on $\mathbb{C}_1^2 \otimes \mathbb{C}_k^2$ and as the identity on the remaining factor $\mathbb{C}_2^2, L_{1k}(\lambda) = \mathscr{P}_{12} L_{2k}(\lambda) \mathscr{P}_{12}$.

Taking into account (9.5) one can see that the R-matrix (9.29) coincides with the L-matrix (9.26) up to a shift of the spectral parameter

$$R(\lambda) = L(\lambda + \frac{\eta}{2}). \tag{9.30}$$

Then by trivially shifting the spectral parameters λ and μ in (9.28) we obtain the Yang–Baxter equation (YBE) [3]

$$R_{12}(\lambda-\mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda-\mu) \;. \tag{9.31}$$

This matrix equation is written in the auxiliary space $\mathrm{End}(\mathbb{C}_1^2 \otimes \mathbb{C}_2^2 \otimes \mathbb{C}_3^2)$, and $R_{12} := R \otimes I,\ R_{23} := I \otimes R,\ R_{13} := \mathscr{P}_{12} R_{23} \mathscr{P}_{12}$. The solution (9.29) is called the Yang R-matrix and there is an obvious extension of it to higher dimensional spaces $\mathbb{C}^n \otimes \mathbb{C}^n$ as the $n^2 \times n^2$ matrix $R(\lambda) = \lambda I + \eta \mathscr{P}$ which also satisfies the YBE (9.31).

The commutation relation for the L-matrix (9.28) induces the commutation relations for the monodromy matrix $T(\lambda)$. These latter have the same form [3–7]

$$R_{12}(\lambda-\mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda-\mu), \tag{9.32}$$

where a convenient notation for tensor products is used $T_1(\lambda) := T(\lambda) \otimes I$, $T_2(\mu) = I \otimes T(\mu)$ [3, 4], see also (7.19) and (7.20). One can extract 16 commutation relations for the entries of $T(\lambda)$ (see e.g. [13]). We will use only few of them (the entries 13, 34, and 14) to algebraically construct the eigenvectors of the hamiltonian H_{XXX}:

$$A(\lambda)B(\mu) = f(\lambda-\mu)B(\mu)A(\lambda) + g(\lambda-\mu)B(\lambda)A(\mu), \tag{9.33}$$
$$D(\lambda)B(\mu) = f(\mu-\lambda)B(\mu)D(\lambda) + g(\mu-\lambda)B(\lambda)D(\mu), \tag{9.34}$$
$$B(\lambda)B(\mu) = B(\mu)B(\lambda), \tag{9.35}$$

where $f(\lambda-\mu) = (\lambda-\mu-\eta)/(\lambda-\mu), g(\lambda-\mu) = \eta/(\lambda-\mu)$. Multiplying the RTT-relation (9.32) by $R_{12}^{-1}(\lambda-\mu)$ and taking the trace over two auxiliary spaces one gets the commutativity property of the transfer matrix $t(\lambda)$,

$$t(\lambda) := \mathrm{tr}\, T(\lambda) = A(\lambda) + D(\lambda), \quad t(\lambda)t(\mu) = t(\mu)t(\lambda). \tag{9.36}$$

The transfer matrix $t(\lambda)$ is a generating function of integrals of motion. Due to the regularity property of the Yang R-matrix

$$R(\lambda;\eta)\big|_{\lambda=0} = \eta\mathscr{P} \tag{9.37}$$

(that in terms of the L-matrix reads $L(\lambda)\big|_{\lambda=\eta/2} = \eta\mathscr{P}$, where $\mathscr{P}$ is the permutation operator (cf. (9.5))) we have that $t(\lambda)\big|_{\lambda=\eta/2}$ is proportional to the shift operator U (9.12). Using the obvious property $\frac{d}{d\lambda}L(\lambda) = I$ it can then further be shown that the logarithmic derivative of $t(\lambda)$ at the point $\lambda = \eta/2$ yields the hamiltonian,

$$H_{XXX} \simeq \frac{d}{d\lambda}\log t(\lambda)\big|_{\lambda=\eta/2}, \tag{9.38}$$

where $\simeq$ stands for equality up to a proportionality factor and a constant additive term (proportional to N). The transfer matrix $t(\lambda)$ is the generating function of the mutually commuting integrals of motions $\mathscr{I}_n = \frac{d^n}{d\lambda^n}\log t(\lambda)\big|_{\lambda=\eta/2}$. These integrals are local densities (a natural and desirable physical property) in the sense that $\mathscr{I}_n$ is a sum of operators each of which acts nontrivially at no more than $n+1$ neighboring sites of the lattice.

We have seen that the hamiltonian can be written in terms of the $A(\lambda)$ and $D(\lambda)$ operators. On the other hand the operators $B(\lambda)$, for different values of λ, generate the eigenvectors of the hamiltonian. They act on the vacuum state (the highest weight vector) Ω defined in (9.8):

$$\Omega = \bigotimes_{1}^{N} e_m^{(+)}, \quad \sigma_m^z e_m^{(\pm)} = \pm e_m^{(\pm)}, \quad \sigma_m^+ e_m^{(+)} = 0, \quad \sigma_m^- e_m^{(+)} = e_m^{(-)},$$

as creation operators for magnons. In order to show that they are creation operators we first observe that

$$C(\lambda)\Omega = 0, \quad A(\lambda)\Omega = a_N(\lambda)\Omega, \quad D(\lambda)\Omega = d_N(\lambda)\Omega,$$

where $a_N(\lambda) = (\lambda+\frac{\eta}{2})^N$, $d_N(\lambda) = (\lambda-\frac{\eta}{2})^N$. This follows from the upper triangular form of the L-matrix on Ω and by recalling the expression of the monodromy matrix $T(\lambda)$ in terms of the $L(\lambda)$ matrices. Next, from the quadratic relation (9.33) for $A(\lambda)$ and $B(\mu)$, we have

$$\begin{aligned} A(\lambda)\prod_{j=1}^{M}B(\mu_j) &= \prod_{j=1}^{M} f(\lambda-\mu_j)B(\mu_j)A(\lambda) \\ &\quad + \sum_{k=1}^{M} g(\lambda-\mu_k)B(\lambda)\prod_{j\neq k}^{M} f(\mu_k-\mu_j)B(\mu_j)A(\mu_k), \end{aligned} \tag{9.39}$$

and a similar relation for $D(\lambda)$ and the product of $B(\mu_j)$. The sum of these relations acting on the vacuum Ω gives the eigenvector (9.16) of the transfer matrix $t(\lambda)$

$$\Psi(\{\mu_j\}_1^M) = \prod_{j=1}^{M} B(\mu_j)\Omega,$$

$$t(\lambda)\prod_{j=1}^{M} B(\mu_j)\Omega = \Lambda(\lambda|\{\mu_k\}_1^M)\prod_{j=1}^{M} B(\mu_j)\Omega, \tag{9.40}$$

under the condition that the parameters μ_k satisfy the Bethe equations $(k = 1,\dots,M)$

$$\frac{a_N(\mu_k)}{d_N(\mu_k)} = \prod_{j\neq k}^{M} \frac{f(\mu_j - \mu_k)}{f(\mu_k - \mu_j)}. \tag{9.41}$$

This condition yields the vanishing of "unwanted terms" containing the operator $B(\lambda)$ and the operators $A(\mu_k), D(\mu_j)$ that as a result of the commutation relations (9.39) have arguments different from μ_j and λ, respectively.

The eigenvalue of the transfer matrix $t(\lambda)$ is

$$\Lambda(\lambda|\{\mu_k\}_1^M) = a_N(\lambda)\prod_{j=1}^{M} f(\lambda-\mu_j) + d_N(\lambda)\prod_{j=1}^{M} f(\mu_j-\lambda).$$

This construction of the eigenvectors of quantum integrable models was coined as the algebraic Bethe ansatz [3].

We conclude by observing that the eigenstates $\Psi = \prod_{j=1}^{M} B(\mu_j)\Omega$, $M \leq [N/2]$ (where $[N/2]$ stands for integer part of $N/2$) are highest weight vectors for the global symmetry algebra $sl(2)$ with generators $S^\alpha = \frac{1}{2}\sum_{n=1}^{N} \sigma_n^\alpha$ (cf. (9.9)),

$$S^+\Psi(\mu_1,\dots,\mu_M) = 0\,, \quad S^z\Psi(\mu_1,\dots,\mu_M) = \left(\frac{N}{2} - M\right)\Psi(\mu_1,\dots,\mu_M)\,. \tag{9.42}$$

The proof is purely algebraic and follows from the RTT-relation and the asymptotic behaviors of the monodromy matrix and of the R-matrix [18],

$$T(\lambda) = \lambda^N I + \eta\lambda^{N-1}\sum_\alpha \sigma_a^\alpha \otimes S^\alpha + O(\lambda^{N-2}), \tag{9.43}$$

$$R_{12}(\lambda-\mu) \simeq I + \frac{\eta}{2\lambda}\left(\sum_\alpha \sigma_1^\alpha \otimes \sigma_2^\alpha + I\right) + O\left(\frac{1}{\lambda^2}\right). \tag{9.44}$$

Indeed, substituting these two asymptotics into the RTT-relation one gets

$$\left[\left(\left(1+\frac{\eta}{2\lambda}\right)I + \frac{\eta}{\lambda}\sum_\alpha \sigma_1^\alpha \otimes \left(\frac{1}{2}\sigma_2^\alpha \otimes 1 + 1 \otimes S^\alpha\right)\right), T_2(\mu)\right] = 0,$$

or

$$\frac{1}{2}\,[\sigma^\alpha, T(\mu)] = [T(\mu), S^\alpha]\,, \quad \frac{1}{2}\,[\sigma^\alpha, T(\mu)]_{xy} = [T(\mu)_{xy}, S^\alpha]\,.$$

The LHS is the commutator of 2×2 matrices while the RHS is the 2×2 matrix of the commutators of the entries of $T(\mu)$ with the global spin generators, e.g.,

$$[S^z, B(\mu)] = -B(\mu), \qquad \left[S^+, B(\mu)\right] = \frac{1}{2}\left(A(\mu) - D(\mu)\right).$$

These relations and (9.33) and (9.34) permit to prove the property (9.42) provided the Bethe equations (9.41) are valid.

9.2.1.1 The Yangian $\mathscr{Y}(sl(2))$

Consider the entries of the 2×2 monodromy matrix $T(\lambda)$ as abstract operators obeying the RTT-relation, divide them by λ^{-N}, and let N be arbitrarily big. We denote by $\mathscr{T}(\lambda)$ this series of 2×2 matrices, with coefficients $t_{ij}^{(n)}$ as abstract generators

$$\mathscr{T}(\lambda)_{ij} = \sum_{n=0}^{\infty} t_{ij}^{(n)} \frac{1}{\lambda^n}, \quad t_{ij}^{(0)} = \delta_{ij}. \tag{9.45}$$

The RTT-relation (9.32) for $\mathscr{T}(\lambda)$ defines an infinite-dimensional Hopf algebra, the Yangian $\mathscr{Y}(gl(2))$. One can define a q-determinant of the matrix $\mathscr{T}(\lambda)$, it is central in $\mathscr{Y}(gl(2))$ and setting it to 1 gives the Yangian $\mathscr{Y}(sl(2))$. The Yangian's coproduct $\Delta : \mathscr{Y}(sl(2)) \to \mathscr{Y}(sl(2)) \otimes \mathscr{Y}(sl(2))$ on the generators $t_{ij}^{(n)}$ can be written in a compact matrix form [19, 20]

$$\Delta(\mathscr{T}(\lambda)_{ij}) = \sum_k \mathscr{T}(\lambda)_{ik} \otimes \mathscr{T}(\lambda)_{kj}. \tag{9.46}$$

According to (9.43) the first nontrivial term $t_{ij}^{(1)}/\lambda$ yields generators of the Lie algebra $sl(2)$ and their coproduct is primitive

$$\Delta(S^\alpha) = S^\alpha \otimes 1 + 1 \otimes S^\alpha.$$

Hence, the universal enveloping algebra $\mathscr{U}(sl(2))$ is a Hopf subalgebra of the Yangian $\mathscr{U}(sl(2)) \subset \mathscr{Y}(sl(2))$. This embedding permits to use twist elements found in $\mathscr{U}(sl(2))^{\otimes 2}$ to perform twisting also of the Yangian (see below and [21]). The Yangian $\mathscr{Y}(g)$ of a Lie algebra g is a deformation of the Lie algebra of polynomial maps $\mathbb{C} \to g$ (or the current algebra $g[t]$), it can also be considered as a "degenerate" version of the quantum affine algebra $\mathscr{U}_q(\hat{g})$, this is a deformation of the central extension $\widehat{L(g)}$ of the loop algebra $L(g)$ (the current algebra $g[t,t^{-1}]$) [9, 20].

9.2.1.2 Higher spins and generalizations

One can take as L-operator the expression similar to (9.26) with an arbitrary representation s_k^α of spin s ($s = 1, 3/2, \ldots$) instead of σ_k^α [7]

$$L_{ak}(\lambda) = \lambda I + \frac{1}{2}\eta \sum_{\alpha} \sigma_a^{\alpha} \otimes s_k^{\alpha}. \tag{9.47}$$

The main QISM relation (9.28) will be still valid with the same 4×4 R-matrix (9.29). This gives us a generalization of the spin 1/2 XXX model to higher spins, i.e., the isotropic spin s model XXX_s [7].

More generally we can consider a solution $R(\lambda;\eta)$ of the YBE (9.31) which has the regularity property $R(\lambda;\eta)\big|_{\lambda=\lambda_0} = \eta \mathscr{P}$ for some value $\lambda = \lambda_0$ (cf. (9.37)) and construct a corresponding quantum integrable system. As before we define the monodromy matrix $T(\lambda)$ as an ordered product of R-matrices (that are related to L-matrices via a formula similar to (9.30)), then the first logarithmic derivative of $t(\lambda)$ gives the hamiltonian H of a spin model

$$H \simeq \frac{d}{d\lambda} \log t(\lambda)\big|_{R(\lambda)=R(\lambda_0)}, \tag{9.48}$$

where, similarly to (9.6),

$$H \simeq \sum_{n=1}^{N} \check{R}_{nn+1} \tag{9.49}$$

and

$$\check{R}_{nn+1} = \mathscr{P}_{nn+1} \frac{d}{d\lambda} R(\lambda)_{nn+1}\big|_{\lambda=\lambda_0}. \tag{9.50}$$

Higher logarithmic derivatives of $t(\lambda)$ give mutually commuting integral of motions.

For the XXX and XXZ models, with chains carrying an arbitrary representation of spin s ($s = 1, 3/2, \ldots$), the constant $\check{R}_{nn+1}$ matrix (9.50) (that is proportional to the permutation matrix P_{nn+1} in the XXX model) satisfies the YBE in the braid group form

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}. \tag{9.51}$$

9.2.2 *Anisotropic* XXZ *spin chain*

The QISM approach to the XXZ model is almost identical to the one discussed for the XXX model. This is so because the corresponding L-matrices and the R-matrices have the same structure and satisfy the same fundamental relations (9.28) and (9.31). We explicitly have

$$L_{XXZ}(\lambda) = \begin{pmatrix} \sinh(\lambda + \eta\sigma_k^{\alpha}/2) & \sinh(\eta)\,\sigma_k^{-} \\ \sinh(\eta)\,\sigma_k^{+} & \sinh(\lambda - \eta\sigma_k^{\alpha}/2) \end{pmatrix}, \tag{9.52}$$

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \tag{9.53}$$

where the entries of the R-matrix are

$$a(\lambda) = \sinh(\lambda + \eta), \quad b(\lambda) = \sinh(\lambda), \quad c(\lambda) = \sinh(\eta).$$

The hamiltonian of the XXZ model is given in (9.20). As in the XXX model the ferromagnetic state Ω is the highest eigenstate of H_{XXZ}, and the L-operator (9.52) and the monodromy matrix $T(\lambda)$ on Ω have an upper triangular structure. Hence the eigenstates, the Bethe equations (9.24), and the energy spectrum are produced by the same algebraic procedure (algebraic Bethe ansatz) that consists of creating magnon states by applying to Ω products of the mutually commuting operators $B(\mu_j)$.

From the quantum group point of view it is more convenient to consider a non-symmetric R-matrix instead of (9.53),

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c_+(\lambda) & 0 \\ 0 & c_-(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \qquad c_\pm(\lambda) = \exp(\pm\lambda)\sinh\eta. \tag{9.54}$$

It is useful to prove directly that due to the commutativity of the R-matrix (9.53) with the primitive coproduct of the Cartan generator $h = \sigma^z$, $[R(\lambda),\ h\otimes 1 + 1\otimes h] = 0$, the transformed R-matrix

$$\exp(x\lambda\ h_1)R_{12}\exp(-x\lambda\ h_1)$$

(where $h_1 = h\otimes 1$, $h_2 = 1\otimes h$) satisfies the YBE (9.31). To obtain (9.54) set $x = \frac{1}{2}$.

The R-matrices (9.53) and (9.54) give the same XXZ model with periodic boundary conditions, but, as we now explain, it is the R-matrix (9.54) that is relevant for the XXZ model with open boundary conditions and that is directly related to the quantum algebras $\mathscr{U}_q(sl(2)) \subset \mathscr{U}_q(\widehat{sl(2)})$. This relation is given via a linear combination of constant R-matrices

$$R(\lambda;q) = \exp(\lambda)R^{(+)}(q) - \exp(-\lambda)R^{(-)}(q), \quad q = \exp\eta,$$

where $R^{(-)}(q) = (R^{(+)}_{21}(q))^{-1} := \mathscr{P}(R^{(+)}_{12})^{-1}\mathscr{P}$. The constant R-matrix

$$R^{(+)}(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \omega & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad \omega = q - \frac{1}{q} = 2\sinh\eta \tag{9.55}$$

enters the RLL relations defining $\mathscr{U}_q(sl(2))$, they are given in (7.63) and (7.64). These relations can be used to prove the RLL relations (9.28) for the L- and R-matrices with spectral parameter (9.52) and (9.53).

By multiplying the $R^{(+)}$-matrix by the permutation $\mathscr{P}$ one gets the matrices

$$\check{R}_k(q) \equiv \check{R}_{k\,k+1}(q) := \mathscr{P} R^{(+)}_{k\,k+1}(q)\,, \quad k = 1, 2, \ldots, N-1\,. \tag{9.56}$$

They satisfy the braid group relation (9.51) and additionally the quadratic relation [20]

$$\check{R}_k(q)^2 = \left(q - \frac{1}{q}\right) \check{R}_k(q) + I. \tag{9.57}$$

The $N-1$ elements $\check{R}_k(q)$ that satisfy (9.51) and (9.57) are the generators of the Hecke algebra $\mathscr{H}_N(q)$.

According to the theory of quantum groups the Hecke algebra $\mathscr{H}_N(q)$ with generators $\check{R}_k(q)$ (9.56) is the centralizer of the diagonal action of $\mathscr{U}_q(sl(2))$ in the space $\otimes_1^N \mathbb{C}^2$,

$$[\check{R}_k(q), \Delta^N(X)] = 0, \quad X \in \mathscr{U}_q(sl(2))\,,$$

where $\Delta^N(X)$ is understood in the representation space $\otimes_1^N \mathbb{C}^2$, and the diagonal action is given by the N-fold coproduct map[1] $\Delta^N : \mathscr{U}_q(sl(2)) \to \mathscr{U}_q(sl(2))^{\otimes N}$,

$$\Delta^N := (\Delta \otimes id \otimes id \otimes \ldots id)(\Delta \otimes id \otimes \ldots id) \ldots (\Delta \otimes id)\Delta\,. \tag{9.58}$$

Let us now consider the hamiltonian of the XXZ model with open boundary conditions

$$H_{XXZ} = \sum_{k=1}^{N-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \cosh\eta\,(\sigma_k^z \sigma_{k+1}^z - 1)\right) + \sinh\eta\,(\sigma_1^z - \sigma_N^z). \tag{9.59}$$

This open spin chain hamiltonian is explicitly $\mathscr{U}_q(sl(2))$ invariant because its density is a cross Casimir of $\mathscr{U}_q(sl(2))$,

$$c_2^{\otimes}(q) = 2(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+) + \cosh\eta\,\sigma_k^z \sigma_{k+1}^z + \sinh\eta\,(\sigma_k^z - \sigma_{k+1}^z).$$

This expression, in accordance with (9.49), essentially coincides with the Hecke algebra generator $\check{R}_k(q)$ (9.56).

Finally we comment on the difference between open and closed (periodic) boundary conditions for the XXX and XXZ models. In the XXX model the difference between open and closed boundary conditions is given by the element $\check{R}_{N1}^{XXX}(q) = P_{N1} = P_{1N}$, that belongs to the symmetry group $\mathscr{S}_N$, so that also H_{XXX} with periodic boundary conditions is an element of the group algebra $\mathbb{C}[\mathscr{S}_N]$, and we

[1] The map Δ^N is the composition of $N-1$ coproduct maps. For example, for $N=3$ we have $\Delta^3 = (\Delta \otimes id)\Delta$. Coassociativity of Δ (cf. (9.79)) then implies that $\Delta^3 = (id \otimes \Delta)\Delta$; similarly Δ^N is independent from the position of Δ in the tensor products (9.58).

have $\mathscr{Y}(sl(2))$ dynamical symmetry. The situation is different in the XXZ model. Indeed in this case the hamiltonian with periodic boundary condition has together with $\check{R}_{kk+1}(q)$ the summand $\check{R}_{N1}(q)$. This latter addend does not belong to the Hecke algebra. This explains why the open spin chain hamiltonian (9.59) is $\mathscr{U}_q(sl(2))$ invariant while the closed spin chain hamiltonian with periodic boundary condition (9.20) is not.

9.3 Twists and QISM

In this section we consider what kind of changes can be induced in integrable spin chains using twist transformations of the related quantum groups.

We see that twists naturally arise when considering scaling limits, for example, the XXX and XXZ models can be related by two inequivalent elementary scaling transformations, and we propose a treatment of the relations obtained via the second scaling limit in terms of a corresponding twist. This leads to the example of the so-called jordanian twist.

In Sect. 9.3.2 on the other hand we consider an abelian twist and study the changes in the hamiltonian of the XXZ model with periodic boundary conditions under this twist transformation.

Section 9.3.3 first details the relation between quantum groups and integrable systems. We then see how, in the case of open spin chains, the twisting of a quantum group leads to the corresponding twisting of the integrable system. Contrary to the case of closed spin chains considered in Sects. 9.3.1 and 9.3.2, the original open spin chain hamiltonian H and the twisted ones $H^{(t)}$ can be easily compared, they are related by a similarity transformation.

In Sect. 9.3.4 we consider another example of twist (coboundary twist) this is in general a trivial twist. Under scaling transformations these coboundary twist can however become nontrivial, this is yet another way to obtain (extended) jordanian twists and their related integrable systems.

Scaling limit $XXZ \to XXX$. It is easy to get the isotropic XXX spin chain from the anisotropic XXZ one via a scaling limit $\varepsilon \to 0$ of parameters

$$\lambda \to \varepsilon\lambda,\, \eta \to \varepsilon\eta,\, q \to 1+\varepsilon\eta,\, \sinh(\lambda-\eta) \to \varepsilon(\lambda-\eta),\, \cosh\eta \to 1+\frac{1}{2}\varepsilon^2\eta^2. \tag{9.60}$$

The hamiltonians (9.20), eigenvectors, and Bethe equations (9.24) are clearly connected in this limit $\varepsilon \to 0$, as well as R-matrices (9.53) and (9.29),

$$R_{XXZ}(\varepsilon\lambda;\varepsilon\eta) \to \varepsilon(\lambda I + \eta\mathscr{P}) = \varepsilon R_{XXX}(\lambda;\eta), \tag{9.61}$$

and the algebraic Bethe ansatz.

Scaling limit $XXZ \to XXX_\xi$. A nontrivial scaling limit (contraction) of the XXZ model is obtained by applying additionally a similarity transformation with the matrix $M(\xi) = \exp(\xi\sigma^+) \in Mat(\mathbb{C}^2)$

$$M(\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \tag{9.62}$$

to the main objects of the QISM. The YBE is obviously invariant with respect to the factorized similarity transformations of its solution $R \to \mathrm{Ad}M^{\otimes 2}R$ [7]. Then the scaling limit (9.60) with a singular behavior of the parameter ξ with respect to ε: $\xi \to \xi/\varepsilon$ yields a deformed XXX spin chain. One obtains the closed spin chain hamiltonian

$$\mathrm{Ad}M(\xi/\varepsilon)^{\otimes N} H_{XXZ} \to H_{XXX_\xi} := H_{XXX} + \sum_{n=1}^{N} \left(\xi^2 \sigma_n^+ \sigma_{n+1}^+ + \xi(\sigma_n^+ - \sigma_{n+1}^+)\right). \tag{9.63}$$

It is a hamiltonian of a deformed XXX model with ξ as the deformation parameter [21]. The similarity transformation does not change the spectrum of H_{XXZ}. Thus in this limit one produces the standard spectrum of the XXX model, although the hamiltonian is now non-hermitian and it depends on ξ. This change of hermiticity comes from the triangularity of the matrix M. In the scaling limit we get additional degeneracy of the spectrum, and some jordanian cells appear. Here is a two-dimensional example of this phenomenon $(\xi \to \xi/(x_2 - x_1), x_2 \to x_1)$

$$AdM(\xi) \cdot \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \begin{pmatrix} x_1 & \xi(x_2 - x_1) \\ 0 & x_2 \end{pmatrix} \underset{(x_2 \to x_1)}{\longrightarrow} \begin{pmatrix} x_1 & \xi \\ 0 & x_1 \end{pmatrix}.$$

The eigenvector $\binom{1}{0}$ survives, while the second eigenvector becomes an adjoint eigenvector.

After this transformation the limiting R-matrix and L-operators, similarly to the hamiltonian (9.63), have also extra terms

$$\mathrm{Ad}M(\xi/\varepsilon)^{\otimes 2} R_{XXZ}(\varepsilon\lambda; \varepsilon\eta) \to \lambda R(\xi) + \eta \mathscr{P}, \tag{9.64}$$

$$(\mathrm{Ad}M_a(\xi) \otimes \mathrm{Ad}M_k(\xi)) L_{ak}^{(XXZ)}(\lambda) \to L_{ak}^{(XXX)}(\lambda; \xi), \tag{9.65}$$

where

$$\begin{aligned} R(\xi) &= I + \xi\left(\sigma^+ \otimes \sigma^z - \sigma^z \otimes \sigma^+ + \xi \sigma^+ \otimes \sigma^+\right) \\ &= \exp(\xi\sigma^+ \otimes \sigma^z)\exp(-\xi\sigma^z \otimes \sigma^+) \end{aligned} \tag{9.66}$$

and

$$\begin{aligned} L_{ak}^{(XXX)}(\lambda; \xi) &= \lambda I + \frac{\eta}{2} \sum_\alpha (\sigma_a^\alpha \otimes \sigma_k^\alpha) \\ &\quad + \xi(\lambda - \eta/2)\left(\sigma_a^+ \otimes \sigma_k^z - \sigma_a^z \otimes \sigma_k^+ + \xi \sigma_a^+ \otimes \sigma_k^+\right). \end{aligned} \tag{9.67}$$

9.3.1 Jordanian twist

The constant $R(\xi)$-matrix satisfies the YBE without spectral parameter. It is a triangular R-matrix: $R_{12}(\xi)R_{21}(\xi) = 1$. It is an image of a universal R-matrix $\mathscr{R} = \mathscr{F}_{21}\mathscr{F}_{12}^{-1}$ obtained by means of a jordanian twist of the universal enveloping algebra $\mathscr{U}(sl(2))$:

$$\mathscr{F}^{(j)} = \exp\left(\frac{1}{2}h \otimes \ln(1+2\xi X^+)\right), \quad w := \ln(1+2\xi X^+), \tag{9.68}$$

where h, $X^\pm$ are the generators of the Lie algebra $sl(2)$: $[h,X^\pm] = \pm 2X^\pm$, $[X^+,X^-] = h$. Let us write down for completeness the twisted coproduct maps for the generators:

$$\begin{aligned}
\Delta_t(h) &:= \mathscr{F}^{(j)}\Delta(h)(\mathscr{F}^{(j)})^{-1} = h \otimes e^{-w} + 1 \otimes h,\\
\Delta_t(X^+) &= X^+ \otimes 1 + 1 \otimes X^+ + 2\xi X^+ \otimes X^+ = X^+ \otimes e^w + 1 \otimes X^+,\\
\Delta_t(w) &= w \otimes 1 + 1 \otimes w,\\
\Delta_t(X^-) &= X^- \otimes e^{-w} + 1 \otimes X^- + \xi h \otimes h e^{-w} + \xi \left(h - \frac{1}{2}h^2\right) \otimes (e^{-w} - e^{-2w}).
\end{aligned}$$

Introducing the new combination $\tilde{X}^- = X^- - \frac{1}{2}\xi h^2$ one obtains a quasiprimitive coproduct also for $\tilde{X}^-$

$$\Delta_t(\tilde{X}^-) = \tilde{X}^- \otimes e^{-w} + 1 \otimes \tilde{X}^-.$$

In the spin 1/2 representation we have $F^{(j)} = \exp(\xi\sigma^z \otimes \sigma^+)$ and $R_{12}(\xi) = \exp(\xi\sigma^+ \otimes \sigma^z)\exp(-\xi\sigma^z \otimes \sigma^+)$.

The scaling limit procedure $XXZ \to XXX_\xi$ does not lead to fully solve the XXX_ξ model because in this limit many eigenstates of the XXZ model become singular (e.g., $\Omega_- = \otimes_k e_k^{(-)}$). New ones have therefore to be found. The study of this closed spin chain quantum integrable system via its R-matrix (9.64) is nontrivial because the form of (9.64) is more complicated than that of (9.61). In particular the commutation relations among the operators $A(\lambda),\ldots,D(\lambda)$ are more involved. Although the monodromy matrix $T(\lambda)$ still has an upper triangular structure when acting on the ferromagnetic state $\Omega = \otimes_1^N e_k^{(+)}$ (9.8), and therefore the operator $B(\lambda)$ is still a creation operator, the algebraic Bethe ansatz is quite elaborated.

Deformations of integrable spin systems related to higher rank Lie algebras, e.g., $gl(n)$ or Lie superalgebras $gl(m|n)$, can be similarly obtained using extended jordanian twists [22, 23]. In particular, a generalization of the isotropic XXX model to the case of $gl(n)$ is given by the hamiltonian

$$H_{gl(n)} = \sum_{m=1}^{N} \mathscr{P}_{mm+1} = \sum_{m=1}^{N}\sum_{i,j=1}^{n} e_{ij}^{(m)} \otimes e_{ji}^{(m+1)}, \tag{9.69}$$

where $\mathscr{P}_{m\,m+1}$ is the permutation operator of $\mathbb{C}^n_m \otimes \mathbb{C}^n_{m+1}$ while $e^{(m)}_{ij}$ are the basic matrices on $\mathbb{C}^n_m$ (with matrix entries $(e^{(m)}_{ij})_{kl} = \delta_{ik}\delta_{jl}$). An extended jordanian twist, e.g., for $n = 3$, is [22]

$$\begin{aligned}\mathscr{F}^{(j_{ext})} &= \exp(2\xi E_{12} \otimes E_{23} \exp(-w_{13})) \exp\left(\frac{1}{2} h \otimes \ln(1 + 2\xi E_{13})\right), \qquad (9.70)\\ w_{13} &= \ln(1 + 2\xi E_{13})),\end{aligned}$$

where h, E_{ij} are the generators of $sl(3)$, $[h, E_{ij}] = (\delta_{1i} + \delta_{3j})E_{ij}$, $[E_{13}, E_{31}] = h$.

9.3.2 Abelian twist

One can add more parameters to the R-matrix of the XXZ model (9.54) using an abelian twist related to the quantum algebra $\mathscr{U}_q(sl(2)) \subset \mathscr{U}_q(\widehat{sl(2)})$. The generator h of $\mathscr{U}_q(sl(2))$ still has the primitive coproduct:

$$\Delta(h) = h \otimes 1 + 1 \otimes h := h_1 + h_2 \in \mathscr{U}_q(sl(2))^{\otimes 2}.$$

Extending this quantum algebra by a central element κ which has also the primitive coproduct $\Delta(\kappa) = \kappa_1 + \kappa_2$, a twist with the carrier space in abelian Lie subalgebra $\mathbb{C}[\kappa, h] \subset \mathscr{U}_q(gl(2))$ can be constructed (i.e., an abelian twist)

$$\mathscr{F}^{(ab)} = \exp(\theta(\kappa \otimes h - h \otimes \kappa)). \qquad (9.71)$$

The transformation of the universal R-matrix is

$$\mathscr{R}^{(t)} = \mathscr{F}_{21} \mathscr{R} \mathscr{F}_{12}^{-1} = \mathscr{F}_{12}^{-1} \mathscr{R} \mathscr{F}_{12}^{-1}, \qquad (9.72)$$

the last equality is due to the property $\mathscr{F}_{21} = \mathscr{F}_{1\,2}^{-1}$ valid for the twist (9.71). Spin $\frac{1}{2}$ representations with fixed central elements $\kappa = \frac{1}{4}$ for both representation spaces $\mathbb{C}^2 \otimes \mathbb{C}^2$ yield

$$R^{(t)}_{12}(\lambda) = \exp\left(\frac{\theta}{4}(\sigma^z_1 - \sigma^z_2)\right) R_{12}(\lambda) \exp\left(\frac{\theta}{4}(\sigma^z_1 - \sigma^z_2)\right). \qquad (9.73)$$

One explicitly obtains

$$R^{(t)}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b_+(\lambda) & c_+(\lambda) & 0 \\ 0 & c_-(\lambda) & b_-(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \qquad \begin{aligned} b_\pm(\lambda) &= p^{\pm 1} \sinh(\lambda), p = \exp(\theta), \\ c_\pm(\lambda) &= \exp(\pm\lambda) \sinh \eta. \end{aligned} \qquad (9.74)$$

The matrix structure of the $R^{(t)}$-matrix is the same as that of (9.54), just the diagonal elements are different. Similarly the L-operator has the same matrix structure. Hence the algebraic Bethe ansatz is also the same. However, since different functions enter relations (9.33) and (9.34), the result is a change of the Bethe equations (9.24) that now read

$$\left(\frac{1}{p}\frac{\sinh(\mu_j+\frac{1}{2}\eta)}{\sinh(\mu_j-\frac{1}{2}\eta)}\right)^N=\prod_{k\neq j}^{M}\frac{\sinh(\mu_j-\mu_k+\eta)}{\sinh(\mu_j-\mu_k-\eta)}. \tag{9.75}$$

Also the hamiltonian depends on the twist parameter $p=\exp(\theta)$,

$$H_{XXZp}=2\sum_{k=1}^{N}\left(\left(p\,\sigma_k^+\sigma_{k+1}^-+\frac{1}{p}\,\sigma_k^-\sigma_{k+1}^+\right)+\frac{1}{2}\cosh\eta\,(\sigma_k^z\sigma_{k+1}^z-1)\right). \tag{9.76}$$

This is the hamiltonian of the closed XXZ_p spin chain, and it is hermitian for $|p|=1$. References on this model studied as spin chain and as $2d$ classical statistical system (6 vertex model) can be found in [24].

The method of constructing new quantum integrable systems via an abelian twist is quite general. There are quantum integrable spin chains corresponding to higher rank ($r>1$) Lie algebras, e.g., $gl(n)$, or Lie superalgebras, e.g., $gl(m|n)$. Then one has an r-dimensional abelian Lie subalgebra, with generators $\{h_j\}_1^r$, and one can construct an abelian twist with more parameters to deform the spin model [25]

$$\mathscr{F}^{(ab)}=\exp\left(\sum\theta^{ij}h_i\otimes h_j\right). \tag{9.77}$$

This twist element is similar to the one used to construct the θ-deformed Poincaré algebra (see [26] and previous chapters).

9.3.3 Generalities on twist transformations

The algebraic structure underlying the main operators entering the QISM: the R-matrix, the L-operator, and the monodromy matrix $T(\lambda)$, is that of a quantum group. In quantum groups a key role is played by the universal R-matrix $\mathscr{R}$ and by the coproduct map Δ. By representing the universal $\mathscr{R}$-matrix and by using the coproduct map Δ one obtains the R-, L-, and T- operators. By twisting the quantum group coproduct map Δ one obtains a new (twisted) quantum group and can consider the corresponding changes of the R-matrix, the L-operator, and the monodromy matrix $T(\lambda)$ that in turn define a new integrable system.

Given a quasitriangular Hopf algebra $\mathscr{U}(m,\Delta,S,\mathscr{R})$ with product m, coproduct Δ, and antipode S, and a twist $\mathscr{F}\in\mathscr{U}\otimes\mathscr{U}$, the corresponding twisted quasitriangular Hopf algebra has a transformed coproduct map Δ_t, for all $a\in\mathscr{U}$,

$$\Delta_t(a)=\mathscr{F}\Delta(a)\mathscr{F}^{-1} \tag{9.78}$$

(cf. Chap. 8.2.1). Coassociativity of this deformed coproduct, i.e.,

$$(\Delta_t\otimes id)\Delta_t=(id\otimes\Delta_t)\Delta_t \tag{9.79}$$

is implied by the Drinfel'd twist equation

$$\mathscr{F}_{12}(\Delta \otimes id)\mathscr{F} = \mathscr{F}_{23}(id \otimes \Delta)\mathscr{F} . \tag{9.80}$$

The corresponding twist-transformed universal R-matrix is

$$\mathscr{R}^{(t)} = \mathscr{F}_{21}\mathscr{R}\mathscr{F}^{-1}. \tag{9.81}$$

Defining Δ^{op} (and similarly Δ_t^{op}) by $\Delta_{12}^{op}(a) = \Delta_{21}(a)$ for all $a \in \mathscr{U}$, we have, again for all $a \in \mathscr{U}$, $\Delta_t^{op}(a) = \mathscr{F}_{21}\Delta^{op}(a)\mathscr{F}_{21}^{-1}$, and

$$\mathscr{F}_{21}\mathscr{R}\Delta(a)\mathscr{F}^{-1} = \mathscr{F}_{21}\Delta^{op}(a)\mathscr{R}\mathscr{F}^{-1}.$$

These two last relations imply the intertwining relation (for all $a \in \mathscr{U}$)

$$\mathscr{R}^{(t)}\Delta_t(a) = \Delta_t^{op}(a)\mathscr{R}^{(t)}.$$

In order to obtain the R-, L-, and T- operators from the universal R-matrix and the coproduct Δ we consider the universal L-matrix. It is an image of the universal R-matrix in a representation ρ corresponding to an auxiliary space V_a

$$\mathscr{L} = (\rho \otimes id)\mathscr{R}, \quad \text{or} \quad \mathscr{L}^{(t)} = (\rho \otimes id)\mathscr{F}_{21}\mathscr{R}\mathscr{F}^{-1}.$$

The L-matrix of the previous sections is then obtained by representing $\mathscr{L}$ on the vector space V_k. The monodromy matrix T of a chain with N sites

$$T_N = L_{aN}L_{aN-1}\dots L_{a1}$$

can be obtained by the action of the N-fold coproduct $\Delta^N : \mathscr{U} \to \mathscr{U}^{\otimes N}$ as defined in (9.58). In fact taking into account the factorization property of the universal R-matrix [9],

$$(id \otimes \Delta)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{12},$$

we have

$$(id \otimes \Delta^3)\mathscr{R} := (id \otimes \Delta \otimes id)(id \otimes \Delta)\mathscr{R} = \mathscr{R}_{14}\mathscr{R}_{13}\mathscr{R}_{12}\,,$$

hence,

$$\mathscr{T}_N = (\rho \otimes id)\Delta^N\mathscr{R} = (\rho \otimes id)\mathscr{R}_{aN}\mathscr{R}_{aN-1}\cdots\mathscr{R}_{a1} \in \mathrm{End}(V_a) \otimes \mathscr{U}^{\otimes N}\,, \tag{9.82}$$

$$T_N = (id \otimes \rho^{\otimes N})\mathscr{T}_N\,. \tag{9.83}$$

Now we consider twist transformations of the monodromy matrix. From (9.82) we see that it is obtained by twisting the universal $\mathscr{R}$-matrix and the N-fold coproduct Δ^N. From the definition of twisted coproduct we have the relation for 3-fold coproducts, for all $a \in \mathscr{U}$,

$$\Delta_t^3(a) = (\Delta_t \otimes id)\Delta_t(a) = \mathscr{F}_{12}\mathscr{F}_{(12)3}\,\Delta^3(a)\,\mathscr{F}_{(12)3}^{-1}\mathscr{F}_{12}^{-1},$$

here $\mathscr{F}_{(12)3} = (\Delta \otimes id)\mathscr{F}$. The N-fold coproduct $\Delta^N : \mathscr{U} \to \mathscr{U}^{\otimes N}$ is also similarly transformed by the twist

$$\Delta_t^N = \mathscr{F}^{(N)}\Delta^N(\mathscr{F}^{(N)})^{-1}, \tag{9.84}$$

where

$$\mathscr{F}^{(N)} := \mathscr{F}_{12}\mathscr{F}_{(12)3}\cdots\mathscr{F}_{(12\ldots N-1)N},$$

and $\mathscr{F}_{(123)4} = (\Delta^3 \otimes id)\mathscr{F}$, and similarly for all other factors up to $\mathscr{F}_{(12\ldots N-1)N} = (\Delta^N \otimes id)\mathscr{F}$.[2]

It is instructive to prove that the relation $\check{\mathscr{R}}^{(t)} = \mathscr{F}\check{\mathscr{R}}\mathscr{F}^{-1}$, defining the twist-transformed $\check{R}$-matrix (cf. (9.81) and (9.50)), in $\mathscr{U}^{\otimes N}$ reads

$$\check{\mathscr{R}}^{(t)}_{nn+1} = \mathscr{F}_{nn+1}\check{\mathscr{R}}_{nn+1}(\mathscr{F}_{nn+1})^{-1} = \mathscr{F}^{(N)}\check{\mathscr{R}}_{nn+1}(\mathscr{F}^{(N)})^{-1}\ . \tag{9.85}$$

The last equality shows that the operator $\mathscr{F}_{nn+1}$ that defines the similarity transformation $\check{\mathscr{R}}_{nn+1} \to \mathscr{F}_{nn+1}\check{\mathscr{R}}_{nn+1}(\mathscr{F}_{nn+1})^{-1}$, and that is local because it depends on the sites n and $n+1$, can be replaced by the operator $\mathscr{F}^{(N)}$ that is global because it is independent from the positions n and $n+1$.

Equality (9.85) allows to compare the hamiltonian $H^{(t)}$ of an open spin chain described by a twisted quantum group to the untwisted one H. Recalling (9.48) we see that

$$H^{(t)} = \sum_{n=1}^{N-1}\check{R}^{(t)}_{nn+1} = F^{(N)}\left(\sum_{n=1}^{N-1}\check{R}_{nn+1}\right)(F^{(N)})^{-1} = F^{(N)}\,H\,(F^{(N)})^{-1}\ , \tag{9.86}$$

where $H^{(t)}, F_{nn+1}$, and $\check{R}_{nn+1}$ are written in a representation. Contrary to the closed spin chains of Sects. 9.3.1 and 9.3.2, we see that the open spin chain hamiltonian $H^{(t)}$ has the same spectrum as H and that its eigenvectors are transformed via $F^{(N)}$.

9.3.4 Coboundary twists and the jordanian deformation

Coboundary twists are twists constructed with any invertible element u of a Hopf algebra $\mathscr{U}$:

$$\mathscr{F}^{(cob)} = (u \otimes u)\Delta(u^{-1}).$$

The Hopf algebra constructed with a coboundary twist has the coproduct $\widetilde{\Delta} = \mathscr{F}^{(cob)}\Delta(\mathscr{F}^{(cob)})^{-1}$ and is isomorphic (as a Hopf algebra) to the original one. They are in fact related by the similarity transformation $\varphi_u : \mathscr{U} \to \mathscr{U}, \quad a \to uau^{-1}$,

[2] Due to the Drinfel'd twist equation (9.80), the N-fold twist $\mathscr{F}^{(N)}$ admits similar and equivalent factorizations with a different order of the $N-2$ coproduct maps acting on different factors of $\mathscr{F}$.

$$\widetilde{\Delta} \circ \varphi_u = (\varphi_u \otimes \varphi_u) \circ \Delta \ .$$

The universal R-matrix of $\mathscr{U}$ (if $\mathscr{U}$ is quasitriangular) is transformed with this twist just by the similarity transformation

$$\mathscr{R} \to \mathrm{Ad}(u \otimes u)\mathscr{R}.$$

We now exploit the very definition of coboundary twist and concoct a coboundary twist of the Hopf algebra $\mathscr{U}_q(sl(2))$ given by an element $u(q,t) \in \mathscr{U}_q(sl(2))$ (where t is a parameter that we later relate to ξ), such that

$$\mathscr{F}^{(cob)}(q,t) = (u \otimes u)\Delta(u^{-1}) \in \mathscr{U}_q(sl(2)) \otimes \mathscr{U}_q(sl(2))$$

is nonsingular in the limit $q \to 1$, while the corresponding element $u(q,t)$ is singular. This coboundary twist in the $q \to 1$ limit is no more a coboundary and leads to the jordanian twist $\mathscr{F}^{(j)}$. Hence, instead of performing a singular contraction of the XXZ model, one can apply the appropriate twist transformation to the whole QISM machinery of the XXZ model and then consider the limit $q \to 1$. An element $u(q,t)$ with these properties is [27]

$$u(q,t) = \exp_{q^2}\left(\frac{t}{1-q^2}X^+\right), \tag{9.87}$$

where

$$\exp_q(x) := \sum_{n=1}^{\infty} \frac{x^n}{(n)_q!} = \exp\left(\sum_{n=1}^{\infty} \frac{(1-q)^{n-1}x^n}{n\,(n)_q}\right), \quad (\exp_q(x))^{-1} = \exp_{q^{-1}}(-x), \tag{9.88}$$

and $(n)_q := (1-q^n)/(1-q)$, $(n)_q! := (1)_q(2)_q\cdots(n)_q$.

Since the generator X^+ of the quantum algebra $\mathscr{U}_q(sl(2))$ has the following coproduct

$$\Delta(X^+) = X^+ \otimes 1 + K^{-2} \otimes X^+,$$

then the coboundary twist element is

$$\begin{aligned} \mathscr{F}^{(cob)}(q) &= (u(q) \otimes u(q))\Delta(u(q)^{-1}) \\ &= \exp_{q^2}\left(\frac{t}{1-q^2}X^+\right) \otimes \exp_{q^2}\left(\frac{t}{1-q^2}X^+\right) \\ &\qquad \exp_{q^{-2}}\left(-\frac{t}{1-q^2}(X^+ \otimes 1 + K^{-2} \otimes X^+)\right). \end{aligned} \tag{9.89}$$

We now use a functional equation for the q-exponential of a sum of noncommuting arguments. Provided that $yx = qxy$ we have

$$\exp_q(x+y) = \exp_q(x)\exp_q(y).$$

Recalling the $\mathscr{U}_q(sl(2))$ commutation relations $K^{-2}X^+ = q^{-2}X^+K^{-2}$ (cf. (7.39) and (7.40)) we can then factorize the third q-exponential in (9.89). Then the expression for $\mathscr{F}^{(cob)}(q)$ simplifies to

$$\mathscr{F}^{(cob)}(q) = \exp_{q^2}\left(\frac{t}{1-q^2} 1 \otimes X^+\right) \exp_{q^{-2}}\left(-\frac{t}{1-q^2}(K^{-2} \otimes X^+)\right).$$

Using the representation of the q-exponential as standard exponential of the q-dilogarithm (9.88), the realization $K^2 = q^h$, and commutativity of the elements $1 \otimes X^+$, $K^{-2} \otimes X^+$, one can show that there are no singular terms in $\mathscr{F}^{(cob)}(q)$ in the limit $q \longrightarrow 1$. The explicit expression is

$$\lim_{q\to 1} \mathscr{F}^{(cob)}(q) = \exp\left(\sum_{n=1}^{\infty} -\frac{1}{2}h \otimes \frac{(tX^+)^n}{n}\right) = \exp\left(\frac{1}{2}h \otimes \ln(1-tX^+)\right),$$

which gives for $t = -2\xi$ the jordanian twist $\mathscr{F}^{(j)}$ (9.68).

9.4 Conclusions

By transforming a given quantum group with a twist we obtain a new quantum group with universal R-matrix changed according to $\mathscr{F}_{21}\mathscr{R}\mathscr{F}^{-1}$. As a result there is a corresponding change in the integrable model associated with the initial quantum group and its representations. It was demonstrated that depending on the properties of the twist the energy spectrum for closed spin chains can be preserved (XXX_ξ model (9.63)) or changed (asymmetric XXZ_p model (9.76)). In both these cases the structure of the eigenstates is also twist dependent. On the other hand, for an open spin chain the twisting procedure simply generates a similarity transformation of the hamiltonian and its eigenstates.

Finally all the new quantum integrable systems obtained by twisting a given quantum integrable system share the same amount of symmetry as the initial one because the amount of symmetry in a group or in a twisted deformation of the group is the same.

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Chapter 10
The Noncommutative Geometry of Julius Wess

Paolo Aschieri

Julius Wess first work on noncommutative geometry dates June 1989. Since then he gradually became more and more interested and involved in this research field. We would like to describe briefly his interests, motivations, and main contributions, which could be divided into four periods. Therefore, we shall trace a short account of his last 18 years of scientific activity and hence of an approach to the subject that has become a reference point for the scientific community.

Physicists' renewed interest in the late 1980s in noncommutative structures was driven by the emergence of quantum groups [1–3]. These deformed structures arose from solving quantum integrable systems with algebraic methods [4] (see Chap. 9) and were also independently studied in the context of C^*-algebras [5]. They found applications in two-dimensional conformal field theories [6] and in constructing new invariants of knots and links and three-dimensional manifolds [7, 8].

Julius Wess initial studies **(1989–1994)** concentrated on quantum groups. One can recognize three parallel investigations:

- Quantum 2×2 matrices and their Lie algebras [9–11]. These are the easiest instances of quantum groups. These studies lead to a classification of deformations of $GL(2)$, and in particular to the description of the Jordanian deformations that as far as we know was first introduced in [12].
- Differential calculus on quantum planes and groups. Quantum matrix groups can be seen as symmetries of noncommutative planes, as initially advocated by Manin [13]. Wess and Zumino developed this viewpoint and studied the differential calculus on the n-dimensional noncommutative planes covariant under the action of the canonically deformed $GL_q(n)$ quantum group [14]. In [11] the differential geometry of the two parameter deformation of the group manifold $GL_{p,q}(2)$ was studied. Thus the first examples of noncommutative differential geometry were considered. Independent similar results were obtained by Woronowicz [15, 16]. These results developed differential geometry by using methods that are different from those advocated by Connes [17].

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- The construction of the quantum Lorentz Lie algebra as symmetry algebra of the complex quantum spinor space [18, 19]. Then by developing the q-differential calculus on the q-Minkowski space (associated with the quantum spinor space), i.e., by finding the generators of translations (partial derivatives), Ogievetsky, Schmidke, Wess and Zumino [20] were able to construct a q-deformed Poincaré Lie algebra (i.e., to construct a one parameter family $q \in \mathbb{R}$ of Hopf algebras) and study a hermitian Laplace operator.

A second investigation period **(1992–2000)**, overlapping with the previous one, was dedicated to representation theory and noncommutative quantum mechanics. A deformation of the two-dimensional phase space was studied [21]. This deformation is derived from a noncommutative differential calculus on the real line, where the momentum operator is obtained from the partial derivative operator $i\partial_x$. The representation of this deformed Heisenberg algebra shows a lattice-like structure, the eigenvalues of the (normalized) position operator as well as those of the momentum operator being $\pm q^n$, n integer, $q \geq 1$ [21, 22]. Another aspect of this investigation is that once the free Hamiltonian in noncommutative phase space $H = \frac{1}{2}P^2$ is rewritten in terms of the usual canonical variables x, p, with $[x, p] = i\hbar$, we obtain a Hamiltonian $H(x, p)$ describing an interacting system. In this way a dynamical system that may be highly nontrivial if thought in commutative spacetime (and undeformed phase space) becomes simple if analyzed in the noncommutative Hamiltonian quantum mechanics framework [23], see also [24, 25]. We will see that this idea is later applied to noncommutative gauge theories. Similar methods have been applied to study the q-deformed quantum mechanical oscillator and its deformed dynamical symmetry algebra [26, 27].

The natural following step in this research was to study the representation of the q-deformed Poincaré algebra constructed in [20]; this was initiated in [28] and further investigated, in particular, in [29, 30]. There also the noncommuting coordinates of q-Minkowski space are represented, and the eigenvalues of the coordinates operators are found: they are discrete and therefore they show a lattice structure of q-noncommutative spacetime.

A related issue is the deformed statistics of identical particles on noncommutative space, this was investigated in [31, 32].

In the years 1997–1999 it was understood that noncommutative gauge theories arise in field theory limits of M-theory and string theory, see [33] and reference therein. In [33] a map, known as Seiberg–Witten map (SW map), between commutative and noncommutative gauge theories was established, under this map a usual gauge transformation is mapped into a noncommutative gauge transformation, in this way gauge equivalent classes $[A_\mu]$ of the usual gauge potential A_μ on commutative spacetime are mapped into gauge equivalent classes $[\hat{A}_\mu]$ of the gauge potential $\hat{A}_\mu$ on noncommutative spacetime (noncommutativity $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ being related to the nonvanishing of the two form B). Under this mapping a complicated gauge theory action on commutative space can have a much nicer expression (and hence interpretation) if written in terms of the gauge potential $\hat{A}_\mu$; the physical content being the same since only gauge equivalence classes are physical. This is the case, for example, of noncommutative electromagnetism whose action is simply $\int \hat{F}^{\mu\nu}\hat{F}_{\mu\nu}$ and captures the low energy physics on a D-brane in the presence of a B-field. These findings triggered a huge amount of literature on the subject of noncommutative field theories. Noncommutativity in this context is mainly implemented via a star product.

Wess and his group in Munich were independently working on the subject of noncommutative gauge theories and (incorporating the results of [33]) they wrote with Madore a first influential paper on the subject [34]: in noncommutative gauge theories the gauge potential naturally arises from considering covariant noncommutative coordinates (rather than from considering covariant derivatives). Noncommutativity is not limited to the case $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with $\theta^{\mu\nu}$ constant; a SW map is considered for more general commutation relations. This noncommutative gauge theory program developed and widened in a series of papers spanning the years **2000–2004**.

- The SW map was used to consider noncommutative gauge theories with arbitrary gauge group (e.g., $SU(n)$ or $SO(n)$) and matter fields in arbitrary representations, thus overcoming the initial restriction to $U(n)$ gauge theories and to matter fields in the fundamental representation of $U(n)$. This was achieved by allowing the noncommutative fields to be valued in the universal enveloping algebra Ug rather than in the Lie algebra g. Now Ug is infinite dimensional; however, by constraining the noncommutative gauge potentials via the SW map, the degrees of freedom remain the same as those of gauge theories on commutative space [35]. The SW map is also extended to matter fields and calculated by introducing the useful consistency check of closure of noncommutative gauge transformations [36] (see Chap. 4).
- In works with Jurčo and Schupp [37–39] the SW map for arbitrary (x-dependent) noncommutativity $\theta^{\mu\nu}$ is understood by using Kontsevich formality map [40]. The same maps that lead to the construction of a star product associated with any given Poisson tensor are used to construct the noncommutative gauge potential $\hat{A}_\mu$ and gauge transformation $\hat{\lambda}$ from the commutative ones A_μ and λ. By studying global properties of these noncommutative gauge theories, noncommutative line bundles with noncommutative transition functions were formulated [41].
- The SW map method allows to expand order by order in powers of the noncommutativity parameters $\theta^{\mu\nu}$ a noncommutative Yang–Mills theory minimally coupled to noncommutative matter fields in terms of the corresponding commutative fields; the result is the corresponding commutative Yang–Mills theory at zeroth order in $\theta^{\mu\nu}$ plus interaction terms at higher order in $\theta^{\mu\nu}$. These extra interactions, like for example contractions of the kind θFFF, respect usual gauge symmetry and also noncommutative gauge symmetry. The vertices involving the noncommutativity parameters $\theta^{\mu\nu}$ break usual Lorentz invariance because $\theta^{\mu\nu}$ is frozen to a given value (it is not a dynamical field). These theories have been shown to be anomaly free [42] if they are so at zeroth order in $\theta^{\mu\nu}$. Using this SW method, deformations of the standard model have been obtained [43, 44]. These noncommutative actions are presently treated as effective actions; however, their renormalization properties are quite interesting. Pure

noncommutative QED breaks renormalizality in an unexpectedly mild way [45–47], hinting at some hidden symmetries; pure Yang–Mills theories at first order in $\theta^{\mu\nu}$ are one-loop renormalizable and, adding an admissible extra interaction term, this is also the case for Yang–Mills theories based on the standard model gauge group [48, 49]. Phenomenological investigations followed [50, 51].

- Another example of noncommutative space is the κ-Minkowski spacetime. There the commutator $[x^\mu, x^\nu]$ is linear in the coordinates [52, 53]. On this space the κ-Poincaré quantum group [54] acts. Scalar and spinor field actions compatible with this quantum symmetry were constructed in [55]. The SW method of defining the noncommutative fields in terms of the commutative fields allows here too to construct a noncommutative Yang–Mills theory invariant under noncommutative gauge transformations. Then, as explained in [56] and in Chap. 5, the usual Yang–Mills theory, plus new gauge-invariant interaction terms depending on the deformation parameter, is obtained by expanding the noncommutative Yang–Mills action in power series of the deformation parameter and in terms of the commutative fields.

The research period **2004–2007** could be seen as a synthesis of the first period on quantum groups and the third one on noncommutative field theories where noncommutativity is given by a $\star$-product. As in [55, 56], it addresses the issues of spacetime symmetries in noncommutative field theory (a main topic of this book).

The compatibility of the canonical commutation relations $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with a deformed (twisted) Poincaré algebra is discovered in [57] (see Chaps. 7, 8); this conclusion was independently reached in [58], see also [59]. The development of the results of [56] leads to the construction of field theories by implementing deformed symmetry principles. As we explain in Chaps. 1, 3, 8, our deformation of the algebra of diffeomorphisms for Moyal–Weyl noncommutative spacetime $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ ($\theta^{\mu\nu}$ constant) leads to a deformed tensor calculus and to the construction of θ-deformed Einstein general relativity [60]. In [61] the differential geometry of a manifold equipped with an arbitrary $\star$-product induced by a twist is considered and developed using a global, coordinate independent formalism. The corresponding deformed Einstein general relativity is obtained. Similar techniques are then applied in order to construct noncommutative gauge theories [62] (see Chaps. 1, 2), and noncommutative supersymmetric theories [63]. See also [64] for twist techniques in quantum field theory.

Julius Wess passed away while fully immersed in the development of this program. Some of the topics he was pursuing or wanted to pursue together with his collaborators, and that indeed are under investigation, are noncommutative gauge theories from Kaluza–Klein reduction of noncommutative gravity; first-order formalism of noncommutative gravity and its coupling to matter fields, e.g., spinors; conservation laws in noncommutative field theories, and in particular the covariant conservation of the Einstein tensor and of the energy momentum tensor in noncommutative gravity; the study of exact solutions of noncommutative gravity; the study of models of twisted supersymmetric field theories; and the generalization of these noncommutative field theory constructions to the case of a wider class of $\star$-products associated with quasitriangular quantum group symmetries rather than triangular ones.

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